The approximation theorem for the characters of "big groups"

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Ergodic group actions and unitary representations seminar
December 21, 2024

Characters on a group

Let G be a topological group. Recall the notion of a character on G:

Definition

A continuous function χ on group G is called a *(normalized) character* if

- χ is positive-definite, i.e. the matrix $[\chi(g_j^{-1}g_k)]_{j,k=1}^m$ is positive semi-definite for any $g_1, g_2, \ldots, g_m \in G$,
- χ is central, i.e. the equality $\chi(gh) = \chi(hg)$ holds for all $g, h \in G$,
- $\chi(e) = 1$, where e is the identity element of G.

Definition

A character χ is called *indecomposable* if it is an extreme point of the simplex of all characters. In other words, χ is indecomposable if there are no distinct characters χ_1, χ_2 and $\alpha \in (0, 1)$ such that $\chi = \alpha\chi_1 + (1 - \alpha)\chi_2$.

General inductive limits

Let $\{G(n)\}_{n=1}^{\infty}$ be a sequence of *locally compact separable groups* such that $G(n) \hookrightarrow G(n+1)$. Consider the inductive limit

$$\varinjlim G(n) = G.$$

The topology on G is given by the usual inductive limit topology.

Let us record the following simple observation:

Lemma

Any compact subset of G is contained in $G(m)$ for all sufficiently large m.

Unitary representations and matrix elements

From now on we will consider only *continuous* unitary group representations, i.e. continuous group homomorphisms $\Pi\colon H \to U(\mathcal{H})$, where H is a topological group and $U(\mathcal{H})$ is the unitary group of a Hilbert space $\mathcal{H}$. We also denote $\mathcal{H}(\Pi) = \mathcal{H}$.

Let T be a unitary representation of G and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of representations of the groups $\{G(n)\}_{n=1}^{\infty}$.

Convention

Given any $\Xi = \{\xi_1, \dots, \xi_s\} \subset \mathcal{H}(T)$ and $\Xi_n = \{\xi_{1n}, \dots, \xi_{sn}\} \subset \mathcal{H}(T_n)$, $n = 1, 2, \dots$, we will write

$$(T_n, \Xi_n) \to (T, \Xi) \text{ as } n \to \infty,$$

if for any $i, j \in \{1, \dots, s\}$ we have the convergence of matrix elements

$$\langle T_n(g)\xi_{in}, \xi_{jn}\rangle \to \langle T(g)\xi_i, \xi_j\rangle, \;\; n \to \infty,$$

which is uniform on compact subsets of G.

Remark

Clearly, for a fixed $g \in G$ the expression $\langle T_n(g)\xi_{in}, \xi_{jn}\rangle$ makes sense for all sufficiently large n.

Convention

Let T be a unitary representation of the group G. A sequence $\{T_n\}_{n=1}^{\infty}$ of unitary representations of the groups $\{G(n)\}_{n=1}^{\infty}$ *approximates* T if for any finite subset $\Xi \subset \mathcal{H}(T)$ there exist finite subsets $\Xi_n \subset \mathcal{H}(T_n)$ of the same cardinality such that

$$(T_n, \Xi_n) \to (T, \Xi) \text{ as } n \to \infty \tag{1}$$

in the sense define above. In this case we will write $T_n \to T$ as $n \to \infty$.

Examples

- For a unitary representation T of the group G define $T_n = T|_{G(n)}$. Then, $T_n \to T$ as $n \to \infty$;
- Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of representations of the groups $\{G(n)\}_{n=1}^{\infty}$ such that $T_n|_{G(m)} = T_m$ for all n and m with $n > m$. Then, for the inductive limit T we have $T_n \to T$ as $n \to \infty$.

On positive-definite functions

Let φ, $\varphi_1, \varphi_2, \ldots$ be normalized, continuous, positive-definite functions on groups G, $G(1), G(2), \ldots$, respectively. Denote by T, T_1, $T_2, \ldots$ the corresponding cyclic unitary representations obtained via the Gelfand–Naimark–Segal (GNS) construction.

Lemma

If $\varphi_n \to \varphi$ when $n \to \infty$ uniformly on compact subsets of G, then $T_n \to T$ as $n \to \infty$.

A representation-theoretic counterpart of the approximation theorem is the following statement:

Theorem (G. Olshanski, [1, 22.9])

For any irreducible unitary representation T of the group G there is a sequence $\{T_n\}_{n=1}^{\infty}$ of irreducible unitary representations of the groups $G(n)$ which approximates T in the sense of (1).

Main theorem

The goal of this talk is to explain the proof of the following theorem.

Theorem (the approximation theorem, G. Olshanski, [1, 22.10])

For any continuous, normalized, indecomposable, positive-definite function φ on G, there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$, where φ_n is a continuous normalized indecomposable positive-definite function on $G(n)$ such that $\varphi_n \to \varphi$ as $n \to \infty$ uniformly on compact subsets of G.

Remark

In the case when G is a discrete group, the last property is equivalent to the pointwise convergence on G.

This theorem is often useful in problems of the classification of all indecomposable characters (equivalently, II_1-factor representations) of "big groups". It essentially reduces the problem to finding all (continuous) limits of irreducible characters of the groups $G(n)$.

Remark

Note that in the theorem above we do not require φ_n or φ to be central.

Example 1: Thoma theorem

The natural inclusions $\{1,\dots,N\} \hookrightarrow \{1,\dots,N,N+1\}$ induce the following chain of embeddings of symmetric groups:

$$\mathfrak{S}(1) \to \mathfrak{S}(2) \to \dots \to \mathfrak{S}(N) \to \mathfrak{S}(N+1) \to \dots$$

The corresponding inductive limit $\mathfrak{S}(\infty) = \varinjlim \mathfrak{S}(n)$ is called the *infinite symmetric group*. The classical theorem of E. Thoma concerns the classification of all indecomposable characters of the infinite symmetric group.

Theorem (Thoma, 1964; Vershik–Kerov, 1981)

The indecomposable characters of the group $\mathfrak{S}(\infty)$ are the functions of the form

$$\chi_{\alpha,\beta}(\sigma) = \prod_{k\in[\sigma]} \left(\sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k \right), \ \sigma \in \mathfrak{S}(\infty),$$

where $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ and $\beta = \{\beta_i\}_{i=1}^{\infty}$ are two sequences of non-negative real numbers (called *Thoma parameters*) such that

$$\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant 0, \ \beta_1 \geqslant \beta_2 \geqslant \dots \geqslant 0, \ \text{and} \ \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leqslant 1.$$

Here $[\sigma]$ stands for the multiset of cycle lengths of a permutation σ.

Example 1: interpretation of the parameters

Recall that irreducible representations of the symmetric group $\mathfrak{S}(N)$ are parameterized by the *Young diagrams* λ with $|\lambda| = N$ boxes. Equivalently, they are in a one-to-one correspondence with the partitions of N, i.e. sequences $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of positive integers with $\lambda_1 \geqslant \dots \geqslant \lambda_\ell$ and $\lambda_1 + \dots + \lambda_\ell = N$. Denote by χ_λ the corresponding normalized character on $\mathfrak{S}(N)$.

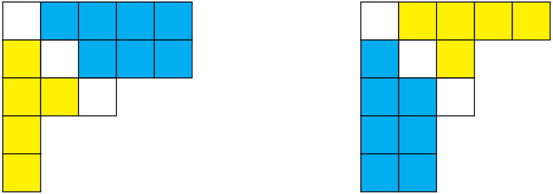
Vershik and Kerov showed that any indecomposable character of $\mathfrak{S}(\infty)$ is a weak limit of a certain sequence $\{\chi_{\lambda(n)}\}_{n=1}^{\infty}$ of irreducible characters of the groups $\mathfrak{S}(n)$. Moreover, a sequence $\{\chi_{\lambda(n)}\}_{n=1}^{\infty}$ converges pointwise iff the following limits exist:

$$\alpha_i := \lim_{n\to\infty} \frac{|i\text{-th row of } \lambda(n)|}{n} \text{ and } \beta_i := \lim_{n\to\infty} \frac{|i\text{-th column of } \lambda(n)|}{n}.$$

Besides that, the corresponding limiting character is precisely the indecomposable character with the Thoma parameters $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_i\}_{i=1}^{\infty}$.

The Thoma parameters correspond to the asymptotics of the lengths of rows and columns of $\lambda(n)$ (or rather as the growth rates of the Frobenius coordinates of $\lambda(n)$).

Young diagrams



Young diagrams for $\lambda = (5,5,3,1,1)$ (left) and $\lambda' = (5,3,3,2,2)$ (right). In the Frobenius coordinates we have $\lambda = (4,3 \mid 4,1)$ and $\lambda' = (4,1 \mid 4,3)$.

Example 2: the infinite unitary group

Similar to the first case one defines the infinite unitary group $\mathrm{U}(\infty) = \varinjlim \mathrm{U}(n)$ using the embedding $U \mapsto \begin{bmatrix} U & \\ & 1 \end{bmatrix}$ for $U \in \mathrm{U}(n)$.

Theorem (Voiculescu, 1976; Vershik–Kerov, 1982)

The indecomposable characters of the group $\mathrm{U}(\infty)$ are the functions of the form

$$\chi_{\alpha^\pm,\beta^\pm,\gamma^\pm}(U) = \det\left(\prod_{k=1}^{\infty} \frac{(I+\beta_k^+(U-I))(I+\beta_k^-(U^*-I))}{(I-\alpha_k^+(U-I))(I-\alpha_k^-(U^*-I))}\right) \times$$
$$\times \exp\left\{\gamma^+ \operatorname{Tr}(U-I) + \gamma^- \operatorname{Tr}(U^*-I)\right\}, \quad U \in \mathrm{U}(\infty),$$

where $\alpha^\pm = \{\alpha_i^\pm\}_{i=1}^\infty$ and $\beta^\pm = \{\beta_i^\pm\}_{i=1}^\infty$ are four sequences of non-negative real numbers, and γ^+, γ^- are non-negative real numbers such that

$$\alpha_1^\pm \geqslant \alpha_2^\pm \geqslant \ldots \geqslant 0,\ \beta_1^\pm \geqslant \beta_2^\pm \geqslant \ldots \geqslant 0,\ \sum_{i=1}^{\infty}(\alpha_i^+ + \alpha_i^- + \beta_i^+ + \beta_i^-) < \infty,\ \beta_1^+ + \beta_1^- \leqslant 1.$$

Example 2: interpretation of the parameters

Recall that the irreducible representations of the group $U(N)$ are parameterized by *highest weights*, i.e. by non-increasing sequences $\lambda = (\lambda_1, \dots, \lambda_N)$ of **integers**. Alternatively, highest weight can be represented by two partitions λ^+ and λ^- which consist of positive entries of $(\lambda_1, \dots, \lambda_N)$ and $(-\lambda_N, \dots, -\lambda_1)$, respectively. Note that $\ell(\lambda^+) + \ell(\lambda^-) \leqslant N$. We abuse the notation and denote by χ_λ the normalized character of the corresponding irreducible representation of $\mathrm{U}(N)$.

Example

For $N = 6$ and highest weight $\lambda = (3, 2, 0, -1, -1, -2)$ we have $\lambda^+ = (3, 2)$ and $\lambda^- = (2, 1, 1)$. In particular, $|\lambda^+| = 5$, $|\lambda^-| = 4$, $\ell(\lambda^+) = 2$ and $= \ell(\lambda^-) = 3$.

It turns out that the approximation theorem holds in the second example as well. Moreover, the parameters $\alpha^\pm$, $\beta^\pm$ and $\gamma^\pm$ have a similar meaning. Namely, a sequence $\{\chi_{\lambda(n)}\}_{n=1}^\infty$ of irreducible characters of the groups $\mathrm{U}(n)$ converges iff the limits

$$\alpha_i^\pm := \lim_{n\to\infty} \frac{|i\text{-th row of } \lambda^\pm(n)|}{n}, \ \beta_i^\pm := \lim_{n\to\infty} \frac{|i\text{-th column of } \lambda^\pm(n)|}{n}, \ \theta^\pm := \lim_{n\to\infty} \frac{|\lambda^\pm|}{n}$$

exist. These correspond to the parameters $\alpha^\pm, \beta^\pm, \gamma^\pm$ in the theorem above, where $\gamma^\pm = \theta^\pm - \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm)$.

About extreme points in locally convex vector spaces

Let L be a real (Hausdorff) locally convex topological vector space. Denote by L^* the space of all continuous linear functionals on L. For a convex subset A of L we denote by $\mathrm{ex}(A)$ the set of all extreme points of A.

The crucial ingredient of the proof of the main theorem is the following general fact.

Proposition ([1, 22.13])

Let $A_1, A_2, \dots$ be a decreasing sequence of convex compact subsets of L that satisfy the first axiom of countability (e.g. metrizable compact subsets). Denote $A = \bigcap_{i=1}^{\infty} A_i$. Then, for any $x \in \mathrm{ex}(A)$ there exists a sequence of points $x_n \in \mathrm{ex}(A_n)$ such that $x_n \to x$ as $n \to \infty$.

One important fact about the extreme points of a convex compact subset is the celebrated Krein–Milman theorem.

Theorem (Krein–Milman)

Let A be a convex compact subset in L. Then, A coincides with the closure of the convex hull of its extreme points. In other words, $A = \overline{\mathrm{conv}(\mathrm{ex}(A))}$.

Auxiliary lemmas

To prove the proposition we need a few auxiliary statements.

Lemma

For any real vector spaces L_1 and L_2 and any convex subsets $B_1 \subset L_1$ and $B_2 \subset L_2$ we have

$$\mathrm{ex}(B_1 \times B_2) = \mathrm{ex}(B_1) \times \mathrm{ex}(B_2).$$

For any $\xi \in L^*$ and $\alpha \in \mathbb{R}$ we put

$$U(\xi, \alpha) = \{y \in L : \xi(y) > \alpha\}, \ V(\xi, \alpha) = \{y \in L : \xi(y) \geqslant \alpha\}.$$

Lemma

Let A be a convex compact subset of L. Let $x \in \mathrm{ex}(A)$ and V be a neighborhood of x in L. Then, there exist $\xi \in L^*$ and $\alpha \in \mathbb{R}$ such that

$$x \in U(\xi, \alpha) \text{ and } A \cap V(\xi, \alpha) \subset V.$$

Proof

The proof follows Dixmier, see [2, Appendix B, B 14]). Note that we may assume that V is open.

(1) By the Hahn–Banach theorem, we have

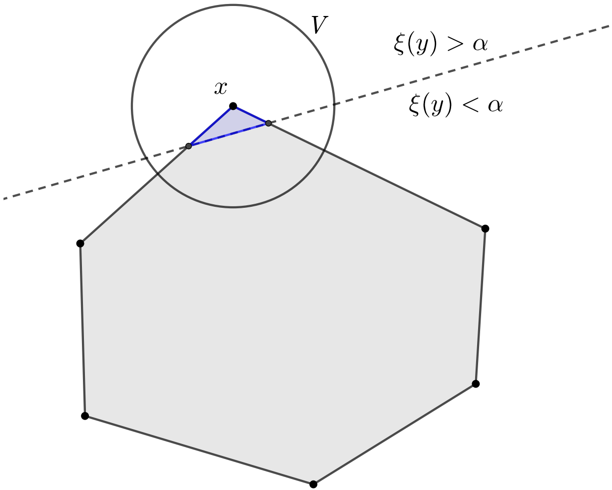
$$A \cap \bigcap_{\substack{\alpha\in\mathbb{R},\, \xi\in L^* \\ \xi(x)>\alpha}} V(\xi,\alpha) = \{x\}, \text{ hence, } (A \setminus V) \cap \bigcap_{\substack{\alpha\in\mathbb{R},\, \xi\in L^* \\ \xi(x)>\alpha}} V(\xi,\alpha) = \varnothing.$$

Note that each $V(\xi,\alpha)$ is a closed subset of L.

(2) Since $A \setminus V$ is compact, we may choose a finite collection of $\xi_1,\dots,\xi_k \in L^*$ and $\alpha_1,\dots,\alpha_k \in \mathbb{R}$ such that $\xi_i(x) > \alpha_i$, that is, $x \in U(\xi_i,\alpha_i)$ for each i and

$$(A \setminus V) \cap \bigcap_{i=1}^{k} V(\xi_i,\alpha_i) = \varnothing.$$

(3) If $k = 1$, then we can take $(\xi,\alpha) = (\xi_1,\alpha_1)$. If $k > 1$, we can decrease k by using the lemma below.


V
$\xi(y) > \alpha$
x
$\xi(y) < \alpha$

Claim

In the notation of the previous lemma assume that $\xi_1, \xi_2 \in L^*$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ are such that $\xi_1(x) > \alpha_1$ and $\xi_2(x) > \alpha_2$. Then, there exist $\xi_3 \in L^*$ and $\alpha_3 \in \mathbb{R}$ such that

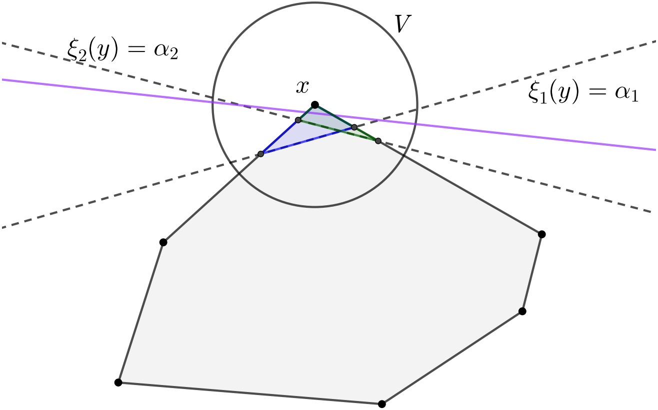
$$\xi_3(\alpha_3) > x \text{ and } A \cap V(\xi_3, \alpha_3) \subset A \cap V(\xi_1, \alpha_1) \cap V(\xi_2, \alpha_2).$$

Proof

Denote $A_i = \{y \in A : \xi_i(y) \leqslant \alpha_i\}$ for $i = 1, 2$. Clearly, A_1 and A_2 are compact convex subsets of A that do not contain x. Moreover, since x is an extreme point of A, the convex hull of A_1 and A_2 does not contain x either. Indeed, we have

$$\operatorname{conv}(A_1 \cup A_2) = \{\lambda a_1 + (1 - \lambda) a_2 : \lambda \in [0, 1]\},$$

and this is a compact convex subset of L that does not contain x. Then, according to the Hahn–Banach theorem, there exist $\xi_3 \in L^*$ and $\alpha_3 \in \mathbb{R}$ such that $\xi_3(x) > \alpha_3$, but $\xi_3(z) < \alpha_3$ for all $z \in \operatorname{conv}(A_1 \cup A_2)$. It is not difficult to see that these ξ_3 and α_3 satisfy the requirements. □


V
$\xi_2(y) = \alpha_2$
x
$\xi_1(y) = \alpha_1$

Main proposition

Recall that L is a locally convex (Hausdorff) topological vector space.

Proposition

Let $A_1, A_2, \ldots$ be a decreasing sequence of compact convex subsets of L that satisfy the first axiom of countability (e.g. metrizable compact subsets). Denote $A = \bigcap_{i=1}^{\infty} A_i$. Then, for any $x \in \operatorname{ex}(A)$ there exists a sequence of points $x_n \in \operatorname{ex}(A_n)$ such that $x_n \to x$ as $n \to \infty$.

Proof

Clearly, it suffices to check that for any neighborhood V of x there is a sequence $\{y_n\}_{n=1}^{\infty}$, where $y_n \in \operatorname{ex}(A_n) \cap V$ for all sufficiently large n. By the above lemma, there exist $\xi \in L^*$ and $\alpha \in \mathbb{R}$ such that $x \in A \cap V(\xi, \alpha) \subset V$.

For any n the set $\{z \in A_n : \xi(z) \geqslant \alpha\}$ is non-empty (it contains x) and thus contains a point $y_n \in \operatorname{ex}(A_n)$. Indeed, by the Krein–Milman theorem, any compact convex subset K of L is the closure of the convex hull of $\operatorname{ex}(K)$. Applying this fact to $\{z \in A_n : \xi(z) \geqslant \alpha\}$ and $\{z \in A_n : \xi(z) = \alpha\}$ gives the required point $y_n \in \operatorname{ex}(A_n)$.

Finally, all limit points of $\{y_n\}_{n=1}^{\infty}$ belong to $A \cap V$ and the lemma follows. □

Towards the proof of the main theorem

To illustrate the main ideas of the proof, we first consider the case where all the groups $G(n)$ and G are discrete.

Let L_n be the space $L^\infty(G(n))$ with respect to the Haar measure endowed with the weak-$*$ topology as the space dual to $L^1(G(n))$.

Introduce the following sets:

- let Q_n be the convex set of all continuous positive-definite functions on $G(n)$ whose values at the identity do not exceed 1;
- let $P_n \subset Q_n$ be the set of normalized *indecomposable* positive-definite functions on $G(n)$ whose values at the identity are equal to 1;
- define P and Q to be the analogous sets for the group G.

As the groups $G(n)$ are separable, the sets Q_n are *metrizable convex compact* subsets of L_n (by the Banach–Alaoglu theorem, the unit ball in $L_n = L^\infty(G(n))$ is compact in the weak-$*$ topology). Note that $\mathrm{ex}(Q_n) = P_n \cup \{0\}$ and $\mathrm{ex}(Q) = P \cup \{0\}$.

Proof in the discrete case 1/2

Let Res_n be the restriction map from functions on $G(n+1)$ to functions on $G(n)$. Observe that since groups $G(n)$ are discrete, the map $\mathrm{Res}_n\colon Q_{n+1} \to Q_n$ is continuous.

Now consider the infinite product $L = L_1 \times L_2 \times \ldots \times L_n \times \ldots$ with the product topology. The space L is locally convex and

$$\widetilde{Q} = Q_1 \times Q_2 \times \ldots \times Q_n \times \ldots$$

is a convex compact metrizable subset of L. We also define

$$A_n = \{f = (f_1, f_2, \ldots) \in L : f_i = \mathrm{Res}_{i+1}(f_{i+1}),\ i = 1, \ldots, n-1\}.$$

It is clear that A_n is a compact convex subset of L that is isomorphic to the product $Q_n \times Q_{n+1} \times \ldots \subset L_n \times L_{n+1} \times \ldots$. Note also that if $f = (f_1, f_2, \ldots) \in \mathrm{ex}(A_n)$, then $f_n \in \mathrm{ex}(Q_n) = P_n \cup \{0\}$. Indeed, this follows from one of the above lemmas if we take $B_1 = Q_n$ and $B_2 = Q_{n+1} \times Q_{n+2} \times \ldots$.

Proof in the discrete case 2/2

Now we may identify Q with $A = A_1 \cap A_2 \cap \ldots$. Then, $\mathrm{ex}(A)$ is identified with $\mathrm{ex}(Q) = P \cup \{0\}$.

Assume that $\varphi \in P$. Then, by the proposition above, there exists a sequence $x_n \in \mathrm{ex}(A_n)$ such that $x_n \to x$ as $n \to \infty$ in the topology of L.

Denote by φ_n the n-th component of $x_n \in A_n$, then $\varphi_n \in \mathrm{ex}(Q_n) = P_n \cup \{0\}$ (recall that $\mathrm{ex}(A_n)$ is identified with $\mathrm{ex}(A_n)$). The definition of the topology of $L = \prod_m L_m$ implies that for any fixed m we have

$$\varphi_n|_{G(m)} \to \varphi|_{G(m)} \text{ as } n \to \infty \text{ in } L_m.$$

As the groups $G(m)$ are assumed to be discrete, we conclude that φ_n converges to φ pointwise on $G(m)$ for each m.

If $\varphi_n \equiv 0$ for infinitely many n, then clearly $\varphi \equiv 0$, which is impossible. Therefore, we may assume that $\varphi_n \in P_n$ for all sufficiently large n. Finally, observe that for discrete groups, the uniform convergence on compact subsets is the same as the pointwise convergence. This concludes the proof of the approximation theorem in the discrete case. $\square$

General case

Now assume that the groups $G(n)$ are arbitrary locally compact separable groups. The main difference is that now the maps $\mathrm{Res}_n \colon L_{n+1} \to L_n$ are no longer continuous and thus, one has to modify the construction of convex compact subsets A_n.

Convention

For two functions f and g on a certain group we write $f \gg g$ if $f - g$ is a positive-definite function.

Let Q_n and L be as above. For each positive integer n define

$$B_n = \{f = (f_1, \dots, f_n) \in Q_1 \times \dots \times Q_n : f_i \gg \mathrm{Res}_i\, f_{i+1} \text{ for } i = 1, \dots, n-1\},$$

$$A_n = B_n \times Q_{n+1} \times Q_{n+2} \times \dots \subset Q_1 \times Q_2 \times \dots \subset L. \quad (2)$$

Clearly, A_n is a convex subset of L.

Lemma

A_n is a metrizable compact subset in the topology of L.

Reminder about group algebras

Let G_0 be a locally compact separable group. Once can regard the elements of the space $L^1(G_0)$ as complex-valued absolutely continuous measures with respect to a left Haar measure ν on G_0. Namely, a function $f \in L^1(G_0)$ corresponds to the measure μ given by

$$\mu(E) = \int_E f(g)\, \mathrm{d}\nu(g),$$

where $E \subset G_0$ is a measurable subset of G_0.

Recall that the Banach algebra $L^1(G_0)$ has a canonical involution which o the level of functions sends $f \in L^1(G_0)$ to

$$\check{f}(g) = \overline{f(g^{-1})}\Delta(g^{-1}),\ g \in G_0,$$

where Δ is the *modular function* of G_0. The latter is defined by means of the identity

$$\mathrm{vol}(Eg^{-1}) = \Delta(g) \cdot \mathrm{vol}(E).$$

Therefore, on the level of measures we have

$$\check{\mu}(E) = \int_E \overline{f(g^{-1})}\Delta(g^{-1})\, \mathrm{d}\nu(g) = \int_{E^{-1}} \overline{f(g)}\, \mathrm{d}\nu(g) = \overline{\mu(E^{-1})}.$$

Convolutions of functions and measures

For two functions $f_1, f_2 \in L^1(G_0)$ we define their convolution as

$$(f_1 * f_2)(g) = \int_{G_0} f_1(h) f_2(h^{-1} g) \, \mathrm{d}\nu(h).$$

In terms of the associated complex-valued measures μ_1 and μ_2 we have

$$(\mu_1 * \mu_2)(E) = \int_{G_0} \int_{G_0} \mathbb{1}_E(gh) \, \mathrm{d}\mu_1(g) \mathrm{d}\mu_2(h),$$

and more generally

$$\int_{G_0} f(k) \, \mathrm{d}(\mu_1 * \mu_2)(k) = \int_{G_0} \int_{G_0} f(gh) \, \mathrm{d}\mu_1(g) \mathrm{d}\mu_2(h).$$

The latter is equivalent to the identity

$$\int_{G_0} f(k) (f_1 * f_2)(k) \, \mathrm{d}\nu(k) = \int_{G_0} \int_{G_0} f(gh) f_1(g) f_2(h) \, \mathrm{d}\nu(g) \mathrm{d}\nu(h).$$

Positive-definite functions and convolutions

Let f be a bounded continuous function on G_0. It is known that f is positive-definite if and only if for any function $p \in L^1(G(i))$ the following inequality holds:

$$\int_{G_0}\int_{G_0} f(g^{-1}h)\overline{p(g)}p(h)\,\mathrm{d}\nu(g)\mathrm{d}\nu(h) \geqslant 0.$$

If μ is the complex-valued measure on G_0 that corresponds to $p \in L^1(G(i))$, then the previous inequality can be rewritten as

$$\int_{G_0}\int_{G_0} f(gh)\overline{p(g)^{-1}}\Delta(g^{-1})p(h)\,\mathrm{d}\nu(g)\mathrm{d}\nu(h) \geqslant 0 \Leftrightarrow \int_{G_0}\int_{G_0} f(gh)\,\mathrm{d}\check{\mu}(g)\mathrm{d}\mu(h) \geqslant 0,$$

that is, $\langle f, \check{\mu} * \mu\rangle = \int_{G_0} f(k)\mathrm{d}(\check{\mu} * \mu)(k) \geqslant 0$.

In other words, f is positive-definite iff $\langle f, \check{\mu} * \mu\rangle \geqslant 0$ for all μ from $L^1(G_0)$.

In particular, if f is a function of the form $\check{q} * F * q$ for some positive-definite function F and $q \in L^1(G_0)$, then f is positive-definite. Indeed, in this case

$$\langle f, \check{\mu} * \mu \rangle = \int_{G_0^4} \check{q}(g_1) F(g_1^{-1} g h h_1) q(h_1^{-1}) \, \mathrm{d}\check{\mu}(g) \mathrm{d}\mu(h) \mathrm{d}\nu(g_1) \mathrm{d}\nu(h_1),$$

which equals $\langle F, (\check{\mu} * \check{q}) * (q * \mu) \rangle = \langle F, \check{r} * r \rangle \geqslant 0$ for $r = q * \mu$.

Therefore,

$$(\check{q} * F * q)(e) = \sup_{\kappa} \langle \check{q} * F * q, \kappa \rangle,$$

where the supremum runs over all probability measures κ from $L^1(g_0)$.

Proof of the lemma

It suffices to check that A_n is closed in L. This can be reduced to verifying that for all i the following set is closed:

$$\{(f_i, f_{i+1}) \in Q_i \times Q_{i+1} : f_i \gg \mathrm{Res}_i\, f_{i+1}\} \subset L_i \times L_{i+1}.$$

Note that we can regard complex-valued measures on $G(i)$ as complex-valued measures on $G(i+1)$ as well.

$$\begin{aligned} f_i \gg \mathrm{Res}_i\, f_{i+1} &\Leftrightarrow \langle f_i, \check{\mu} * \mu\rangle \geqslant \langle \mathrm{Res}_i\, f_{i+1}, \check{\mu} * \mu\rangle \text{ for all } \mu \in L^1(G(i)) \Leftrightarrow \\ &\Leftrightarrow \langle f_i, \check{\mu} * \mu\rangle \geqslant \langle f_{i+1}, \check{\mu} * \mu\rangle \text{ for all } \mu \in L^1(G(i)) \Leftrightarrow \\ &\Leftrightarrow \langle f_i, \check{\mu} * \mu\rangle \geqslant (\check{\mu} * f_{i+1} * \mu)(e) \text{ for all } \mu \in L^1(G(i)). \end{aligned}$$

Observe that the function $\check{\mu} * f_{i+1} * \mu$ is a positive-definite function on $G(i+1)$ and therefore,

$$(\check{\mu} * f_{i+1} * \mu)(e) = \sup_{\kappa} \langle \check{\mu} * f_{i+1} * \mu, \kappa\rangle = \sup_{\kappa} \langle f_{i+1}, \check{\mu} * \kappa * \mu\rangle,$$

where κ in the supremum runs over all probability measures from $L^1(G(i+1))$. Thus, $f_i \gg \mathrm{Res}_i\, f_{i+1}$ holds if and only if $\langle f_i, \check{\mu} * \mu\rangle \geqslant \langle f_{i+1}, \check{\mu} * \kappa * \mu\rangle$ for all μ and κ. Since these conditions are closed in the topology of $L_i \times L_{i+1}$, the lemma follows. □

Towards the proof

For positive integers m and n with $m \leqslant n$ define the following subset of B_n

$$Q_{n,m} = \left\{ f = (f_1, \ldots, f_n) \in Q_1 \times \ldots \times Q_n \;\middle|\; \begin{array}{l} f_i = \mathrm{Res}^{m-i} f_m \text{ for } i = 1, \ldots, m \\ f_i = 0 \text{ for } i = m+1, \ldots, n \end{array} \right\} \subset B_n.$$

For instance, $Q_{4,2} = \{(\mathrm{Res}_1 f, f, 0, 0) : f \in Q_2\}$. Clearly, the mapping $f \mapsto f_m$ gives an isomorphism between Q and Q_m.

For any $g \in Q_m$ we denote by $g^{(n)}$ the corresponding element from $Q_{n,m}$.

Lemma

The set B_n is the convex hull of its subsets $Q_{n,1}, Q_{n,2}, \ldots, Q_{n,n}$.

Proof

It is evident that $\mathrm{conv}(Q_{n,1} \cup Q_{n,2} \cup \ldots \cup Q_{n,m}) \subset B_n$. Take any $f = (f_1, \ldots, f_n) \in B_n$. For each $i = 1, \ldots, n-1$ define an element g_i of L_i by the formula $g_i = f_i - \mathrm{Res}_i f_{i+1}$. By the definition of B_n, we have $g_i \gg 0$ and $g_i(e) = f_i(e) - f_{i+1}(e) \leqslant 1$. Therefore, $g_i \in Q_i$ for every $i = 1, \ldots, n-1$.

Proof

Put $g_n := f_n \in Q_n \subset L_n$. We have the following equalities

$$f_1 = \mathrm{Res}^{n-1}(g_n) + \mathrm{Res}^{n-2}(g_{n-1}) + \ldots + \mathrm{Res}(g_2) + g_1,$$
$$f_2 = \mathrm{Res}^{n-2}(g_n) + \mathrm{Res}^{n-3}(g_{n-1}) + \ldots + g_2,$$
$$\ldots \quad \ldots \quad \ldots$$
$$f_n = g_n$$

In other words, $f = g_n^{(n)} + g_{n-1}^{(n)} + \ldots + g_1^{(n)}$. As $f_1(e) = g_n(e) + g_{n-1}(e) + \ldots + g_1(e)$, the function f is a convex combination of $(a_n g_n)^{(n)}, (a_{n-1}g_{n-1})^{(n-1)}, \ldots, (a_1 g_1)^{(n)}$:

$$f = \frac{g_n(e)}{f_1(e)} \cdot (a_n g_n)^{(n)} + \frac{g_{n-1}(e)}{f_1(e)} \cdot (a_{n-1}g_{n-1})^{(n)} + \ldots + \frac{g_1(e)}{f_1(e)} \cdot (a_1 g_1)^{(n)}, \text{ where}$$
$$a_n = \frac{f_1(e)}{g_n(e)}, a_{n-1} = \frac{f_1(e)}{g_{n-1}(e)}, \ldots, a_1 = \frac{f_1(e)}{g_1(e)}.$$

If $g_i(e) = 0$ for some i, then $g_i \equiv 0$ and we omit the corresponding term. It remains to notice that $a_n g_n(e) = a_{n-1}g_{n-1}(e) = \ldots = a_1 g_1(e) = f_1(e) \leqslant 1$, so $a_i g_i \in Q_i$ for all i. Thus, $(a_i g_i)^{(n)} \in Q_i$ and $f \in \mathrm{conv}(Q_{n,1} \cup \ldots Q_{n,n})$, as claimed. □

Lemma

$\mathrm{ex}(B_n) = \mathrm{ex}(Q_{n,1}) \cup \mathrm{ex}(Q_{n,2}) \cup \ldots \cup \mathrm{ex}(Q_{n,n})$.

Proof

The previous lemma implies that $\mathrm{ex}(B_n) \subset \bigcup_{i=1}^{n} \mathrm{ex}(Q_{n,i})$. The reverse inclusion follows from the fact that the zero function on $G(i)$ belongs to $\mathrm{ex}(Q_i)$. □

Now consider the sets $A_1, A_2, \ldots$ defined in (2) and put $A = \bigcap_{n=1}^{\infty} A_n$. Then,

$$A = \{f = (f_1, f_2, \ldots,) \in Q_1 \times Q_2 \times \ldots : f_i \gg \mathrm{Res}_i\, f_{i+1} \text{ for } i = 1, 2, \ldots\}.$$

In particular, we can regard Q as a subset of A.

Lemma

$\mathrm{ex}(Q) \subset \mathrm{ex}(A)$.

Proof

For any element $f = (f_1, f_2, \ldots)$ in A we have $f_1(e) \geqslant f_2(e) \geqslant \ldots$. It remains to note that the elements of Q can be distinguished by the condition $f_1(e) = f_2(e) = \ldots$, which implies the statement of the lemma. □

Proof of the general case 1/2

Let us show that for any $\varphi \in P$ there exists an increasing sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ and elements $\varphi_{n_j} \in \mathrm{ex}(P_{n_j})$ such that $\varphi_{n_j} \to \varphi$ as $j \to \infty$ uniformly on all compact subsets of G.

Identify $\varphi \in P$ with a point $x \in A$. Then, $x \in \mathrm{ex}(A)$ since $\mathrm{ex}(Q) = P \cup \{0\}$. Recall that $A = \bigcap_{n=1}^{\infty} A_n$ and hence by the proposition above, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to x in L and such that $x_n \in \mathrm{ex}(A_n)$ for all n.

Write $x_n = (f_{n,1}, f_{n,2}, \ldots)$, where $f_{n,i} \in Q_i$. Then, it is not difficult to see that for any fixed m we have

$$f_{n,m} \to \varphi|_{G(m)} \text{ as } n \to \infty \tag{3}$$

in the topology of $L(m)$. Since $\varphi \not\equiv 0$, we have $f_{n,m} \not\equiv 0$ for n which are sufficiently large with respect to m.

On the other hand, the condition $x_n \in \mathrm{ex}(A_n)$ implies that $(f_{n,1}, \ldots, f_{n,n}) \in \mathrm{ex}(B_n)$. Then, one of the lemmas above implies that

$$(f_{n,1}, \ldots, f_{n,n}) \in \mathrm{ex}(Q_{n,k}) \setminus \{0\}$$

for some $k = k(n) \in \{1, \ldots, n\}$.

Proof of the general case 2/2

In particular, we obtain that

$$f_{n,k(n)} \in P_{k(n)} \text{ and } f_{n,i} \equiv 0 \text{ for } i > k(n).$$

It follows from (3) and the fact that $\varphi \not\equiv 0$ that $k(n) \to \infty$ as $n \to \infty$. Moreover, for any fixed m we have the convergence

$$f_{n,k(n)}|_{G(m)} \to \varphi|_{G(m)}$$

in the topology of $L(m)$ as $n \to \infty$. Finally, recall that for normalized positive-definite functions on $G(m)$, the convergence in the topology of the space $L(m)$ coincides with the uniform convergence on compact subsets, see e.g. Dixmier [2, 13.5.2].

Finally, choosing a subsequence $\{n_j\}_{j=1}^{\infty}$ such that $\{k(n_j)\}_{j=1}^{\infty}$ is increasing and setting $\varphi_{n_j} := f_{n_j,k(n_j)}$ gives the required statement. □

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