BIG ALGEBRA IN TYPE A FOR THE COORDINATE RING OF THE MATRIX SPACE

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Abstract. In this note we obtain the explicit formulas for the big algebra generators acting on a polynomial ring. Our approach is based on direct computations in Weyl algebra using Kirillov D-operator, Capelli-type identities. We verify that big algebra in type A is in fact the image of a suitable Bethe algebra.

1. Introduction

In this note we obtain explicit formulas for generators of big algebras in type A embedded into algebra of polynomial differential operators on $\mathfrak{gl}_n$. Using these formulas, the formalism of R-matrices and ternary relations we give an alternative proof of the commutativity of the big algebra. This approach has an advantage of considering big algebras of all polynomial representations of $\mathfrak{gl}_n$ simultaneously. Besides that, our proof of commutativity relies only on direct calculations and thus, avoids the use of non-trivial constructions such as Feigin-Frenkel center and opers.

1.1. **Contents.** Now let us briefly outline the contents of this note.

In Section 2 we fix the notation and recall the necessary facts about the representation $\mathfrak{gl}_n$ on the matrix space $\mathrm{Mat}(n, r)$.

In Section 3 we remind the notions of the classical Kirillov algebra and big algebra. Then, we state the main result of this note – the explicit formulas for the generators of big algebra of the coordinate ring of $\mathrm{Mat}(n, r)$ (Theorem 3.8 and Corollary 3.9).

Section 4 is rather technical and is devoted to proofs of Theorem 3.8 and Corollary 3.9.

Sections $5-7$ contain the proof of the commutativity of big algebras. In Section 5 we revise the Capelli's identity and its variants and in Section 6 we we recall the construction of a certain commutative subalgebra of $U(\mathfrak{gl}_n)$, called *Bethe algebra* following Molev [8, Section 1.14]. Finally, in Section 7 we prove the commutativity of big algebra (in type A) using the explicit formulas obtained in Section 3 (Corollary 3.9) and the results from Sections 5 and 6.

2. Notation and preliminaries

Most of the proofs in this note involve many direct calculations. To simplify the formulas we introduce the following notations.

2.1. **Operations with tuples.** For every positive integer m we denote $[m] := \{1, \dots, m\}$ and let $\mathfrak{S}_m$ be the symmetric group of $[m]$. For any integer k such that $0 \le k \le m$ we define $\binom{[m]}{k}$ to be the set of all k-element subsets of $[m]$ and $[m]^{\underline{k}}$ to be the set of all k-tuples which consist of k distinct elements of m. Clearly,

$$\#\binom{[m]}{k} = \binom{m}{k}, \ \#[m]^{\underline{k}} = m^{\underline{k}},$$

where $m^{\underline{k}}$ is the so-called *falling factorial*:

$$m^{\underline{k}} = m(m-1)\dots(m-k+1) = k! \cdot \binom{m}{k}.$$

We also denote by $[m]^k = [m]^{[k]}$ the set of all k-tuples with entries in $[m]$. It is occasionally convenient to view a k-tuple $I = (i_1, \dots, i_k)$ as a function on the set $[k] = \{1, \dots, k\}$, namely, we set $I(l) = i_l$ for $l \in [k]$.

2.1.1. *Action of the symmetric group.* For any k-tuple $I = (i_1, \dots, i_k)$ and any permutation $\pi \in \mathfrak{S}_k$ define

$$\pi I = (i_{\pi^{-1}(1)}, \dots, i_{\pi^{-1}(k)}). \tag{2.1}$$

Regarding I as a funciton on $[k]$ we can alternatively write $\pi I = I \circ \pi^{-1}$.

Remark 2.1. Using this group action we can identify $\binom{[m]}{k}$ with quotient $\mathfrak{S}_k \backslash [m]^{\underline{k}}$. In particular, we will often regard a k-element subset as a k-tuple with arbitrarily chosen ordering (and in these situations the choice of ordering will be irrelevant).

Date: September 14, 2024.

2.1.2. *Sign functions for tuples.* For any $I, J \in [n]^{\underline{k}}$ define the generalized sign function as follows:

$$\operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix} = \begin{cases} \operatorname{sgn}(\tau), & \text{if as sets } I = J, \\ 0, & \text{otherwise,} \end{cases}$$

where in the first case τ is the unique permutation in $\mathfrak{S}_k$ that maps k-tuple J to I. In other words, τ satisfies $I = \tau \cdot J$.

Remark 2.2. Clearly, this agrees with the usual sign of permutation in the case when both I and J are permutations of $(1, 2, \ldots, n)$. In particular, if $I = (1, \ldots, n)$ and $J = (\tau(1), \ldots, \tau(n))$ for $\tau \in \mathfrak{S}_n$, then

$$\operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \ldots & n \\ \tau(1) & \tau(2) & \ldots & \tau(n) \end{pmatrix} = \operatorname{sgn}(\tau).$$

Now assume that we have tuples $I_1, \ldots, I_k$ and $J_1, \ldots, J_k$ such that $I_l, J_l \in [n]^{\underline{p_l}}$ for each $l = 1, \ldots, k$ and some positive integers $p_1, \ldots, p_k$. Denote also $p = p_1 + \ldots + p_k$. Then, we define

$$\operatorname{sgn}\begin{pmatrix} I_1 & \ldots & I_k \\ J_1 & \ldots & J_k \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix},$$

where I and J are p-tuples obtained by concatenating $I_1, \ldots, I_k$ and $J_1, \ldots, J_k$, respectively. For example, if $k = 2$, then

$$I(s) = \begin{cases} I_1(s), & s \in \{1, \ldots, p_1\}, \\ I_2(s - p_1), & s \in \{p_1 + 1, \ldots, p_1 + p_2\}, \end{cases} \quad \text{and } J_1(s) = \begin{cases} J_1(s), & s \in \{1, \ldots, p_1\}, \\ J_2(s - p_1), & s \in \{p_1 + 1, \ldots, p_1 + p_2\}. \end{cases}$$

In our calculations we also need another variant of the signature function. For any tuples $I_1, J_1 \in [m]^{\underline{p}}$ and $I_2, J_2 \in [m]^{\underline{q}}$ define

$$\varepsilon(I_1, J_1, I_2, J_2) = \begin{cases} \operatorname{sgn}(\tau_1 \tau_2), & \text{if as sets } I_1 \setminus I_2 = J_1 \setminus J_2 \in \binom{[m]}{p-q}, \\ 0, & \text{otherwise.} \end{cases}$$

Here in the first case τ_1 and τ_2 are elements of $\mathfrak{S}_p$ that satisfy equalities:

$$\tau_1 I_1|_{[q]} = J_1, \;\; \tau_2 I_2|_{[q]} = J_2, \;\text{ and } \tau_1 I_1|_{\{p+1,\ldots,q\}} = \tau_2 I_2|_{\{p+1,\ldots,q\}}.$$

Note that a pair (τ_1, τ_2) is not defined uniquely in general. However, for any other such pair (τ_1', τ_2') there exists an element $\sigma \in \mathfrak{S}_p$ which fixes each element of $[q]$ and such that $(\tau_1', \tau_2') = (\sigma\tau_1, \sigma\tau_2)$. In particular, $\operatorname{sgn}(\tau_1\tau_2) = \operatorname{sgn}(\tau_1'\tau_2')$ and, consequently, $\varepsilon(I_1, J_1, I_2, J_2)$ is well defined.

Observe that both generalized sign functions are skew-symmetric in the sense of the following lemma.

Lemma 2.1. *For any tuples $I, J \in [m]^{\underline{p}}$ and any permutation $\sigma, \tau \in \mathfrak{S}_p$ and one has*

$$\operatorname{sgn}\begin{pmatrix} \sigma I \\ \tau J \end{pmatrix} = \operatorname{sgn}(\sigma\tau) \operatorname{sgn}\begin{pmatrix} I \\ J \end{pmatrix}.$$

Similarly, for ay $I_1, J_1 \in [m]^{\underline{p}}$ and $I_2, J_2 \in [m]^{\underline{q}}$ and any permutations $\sigma_1, \tau_1 \in \mathfrak{S}_p$ and $\sigma_2, \tau_2 \in \mathfrak{S}_q$ one has

$$\varepsilon(\sigma_1 I_1, \tau_1 J_1, \sigma_2 I_2, \tau_2 J_2) = \operatorname{sgn}(\sigma_1\tau_1) \operatorname{sgn}(\sigma_2\tau_2) \cdot \varepsilon(I_1, J_1, I_2, J_2).$$

2.2. **Non-commutative matrices.** We often work with matrices whose elements are elements of non-commutative algebras. In particular, many computations involve non-commutative versions of determinants.

Let $A = [a_{ij}]_{i,j=1}^N$ be an $N \times N$ matrix with entries in a certain non-commutative algebra. We define for matrix A its

- *row determinant*:
$$\operatorname{rdet}(A) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{N,\sigma(N)},$$
- *column determinant*:
$$\operatorname{cdet}(A) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \ldots a_{\sigma(N),N},$$
- *symmetrized determinant*:
$$\operatorname{symdet}(A) = \sum_{\sigma,\tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau) a_{\sigma(1),\tau(1)} a_{\sigma(2),\tau(2)} \ldots a_{\sigma(N),\tau(N)}.$$

Observe that if the entries of A do commute, then row and column determinants coincide with the usual one while for the symmetrized version one has $\operatorname{symdet}(A) = N! \cdot \det(A)$. Finally, note that row (column) determinant is skew-symmetric with respect to columns (rows) while the symmetrized determinant is skew-symmetric with respect to both rows and columns.

For any $N\times N$ matrix M and any tuples $I=(i_1,\dots,i_k)\in[N]^k$ and $J=(j_,\dots,j_l)\in[N]^l$ we denote by M_{IJ} the following $k\times l$ matrix:

$$M_{IJ}=[M_{i_\alpha,j_\beta}]_{\alpha\in[k],\beta\in[l]}=\begin{bmatrix}M_{i_1j_1} & \dots & M_{i_1j_l}\\ \vdots & \ddots & \vdots\\ M_{i_kj_1} & \dots & M_{i_kj_l}\end{bmatrix}.$$

In the case, when the entries of I and J are strictly increasing, M_{IJ} is a submatrix of M.

2.3. **Lie algebra $\mathfrak{gl}_n$.** Let $\mathfrak{g}=\mathfrak{gl}_n$ be the general Lie algebra of complex $n\times n$ matrices. We denote by $\{E_{ij}\}_{i,j=1}^n$ the standard basis of $\mathfrak{gl}_n$ consisting of matrix units which satsify the following commutation relations:

$$[E_{ij},E_{kl}]=\delta_{jk}E_{il}-\delta_{li}E_{kj}. \tag{2.2}$$

Denote also by $\{y_{ij}\}_{i,j=1}^n$ the corresponding coordinates on $\mathfrak{gl}_n$. In other words, $y_{ij}\in\mathfrak{gl}_n^*$ and for any $Y\in\mathfrak{gl}_n$ we have

$$Y=\sum_{i,j=1}^n y_{ij}(Y)\cdot E_{ij}.$$

In particular, we can view the algebra $\mathbb{C}[\mathfrak{gl}_n]=S(\mathfrak{gl}_n^*)$ as the polynomial ring $\mathbb{C}[y_{ij},1\le i,j\le n]$.

Let $\mathfrak{h}=\operatorname{span}\{E_{ii}\}_{1\le i\le n}$ be the Cartan subalgebra consisting of diagonal matrices. Let $\mathfrak{n}_+=\operatorname{span}\{E_{ij}\}_{1\le i<j<\le n}$ and $\mathfrak{n}_-=\operatorname{span}\{E_{ij}\}_{1\le j<i<\le n}$ be the nilpotent subalgebras consisting of upper-triangular and lower-triangular matrices, respectively. Thus, we obtain a triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$.

2.4. **Coordinate ring of $n\times r$ matrices.** Consider the coordinate ring $\mathcal{P}(n,r)=\mathbb{C}[\mathrm{Mat}(n,r)]$ of the space $\mathrm{Mat}(n,r)$ of complex $n\times r$ matrices. Denote by $\{x_{ij}:1\le i\le n,1\le j\le r\}$ the standard coordinates on $\mathrm{Mat}(n,r)$. Then, $\mathcal{P}(n,r)$ is the polynomial ring $\mathbb{C}[x_{ij},1\le i\le n,1\le j\le r]$ in rn variables. We denote by ∂_{ij} the partial derivative with respect to variable x_{ij}.

2.4.1. *Action of $\mathfrak{gl}_n$.* Note that the matrix space $\mathrm{Mat}(n,r)$ possesses the following GL_n-action on $\mathrm{Mat}(n,r)$:

$$(g,A)\mapsto(g^{-1})^T\cdot A,\ g\in\mathrm{GL}_n, A\in\mathrm{Mat}(n,r).$$

This action induces a GL_n-action on the coordinate ring $\mathcal{P}(n,r)=\mathbb{C}[\mathrm{Mat}(n,r)]$. Namely, for any $P\in\mathcal{P}(n,r)$ we have

$$(g\cdot P)(A)=P(g^T\cdot A),\ g\in\mathrm{GL}_n, A\in\mathrm{Mat}(n,r).$$

Denote by $\widetilde{L}(g)$ the corresponding linear operator in $\operatorname{End}\mathcal{P}(n,r)$.

Differentiating this action along one-parameter subgroups in GL_n yields the infinitesimal $\mathfrak{gl}_n$-action on $\mathcal{P}(n,r)$, which we denote by L. Direct calculation shows that

$$L(E_{ij})=\sum_{\alpha=1}^r x_{i\alpha}\partial_{j\alpha},\ 1\le i,j\le n.$$

2.4.2. *Algebra $\mathcal{PD}(n,r)$ of differential operators on $\mathcal{P}(n,r)$.* Since L is a representation of $\mathfrak{gl}_n$ it also gives rise to a representation of the universal enveloping algebra $U(\mathfrak{gl}_n)$ on $\mathcal{P}(n,r)$. In particular, the formula for $L(E_{ij})$ above implies that elements of $U(\mathfrak{gl}_n)$ act on $\mathcal{P}(n,r)$ as differential operators with polynomial coefficients.

Let $\mathcal{PD}(n,r)$ be the (non-commutative) algebra of differential operators on $\mathcal{P}(n,r)$ with polynomial coefficients. In other words, $\mathcal{PD}(n,r)$ is the Weyl algebra generated by $\{x_{i\alpha},\partial_{i\alpha}:1\le i\le n,1\le\alpha\le r\}$ and relations of the form

$$[\partial_{i\alpha},x_{j\beta}]=\delta_{ij}\delta_{\alpha\beta},\ 1\le i,j\le n,1\le\alpha,\beta\le r.$$

We will need the following fact in Section 5.

Proposition 2.2. *If $r=n$, then the map $L\colon U(\mathfrak{gl}_n)\to\mathcal{PD}(n,r)$ is injective.*

Remark 2.3. In fact, this follows from the fact that $U(\mathfrak{gl}_n)$ is isomorphic to the algebra of all left-invariant differential operators on GL_n. See also [5, Section 1].

Denote by E, X and D the matrices, whose (i,j)-th entry equals E_{ij}, x_{ij} and ∂_{ij}, respectively. In particular, we have a formal identity

$$L(E)=X\cdot D^T.$$

3. Kirillov algebra, big algebra and medium algebra on $\mathcal{P}(n,r)$

3.1. **Invariant polynomials on $\mathfrak{gl}_n$.** For each $0 \le k \le n$ define the following element of $S(\mathfrak{gl}_n^*)$:

$$c_k(Y) = \operatorname{tr}(\Lambda^k Y),\ Y \in \mathfrak{gl}_n,$$

where $\Lambda^k Y$ is the operator in $\operatorname{End}\Lambda^k(\mathbb{C}^n)$ which is induced by the natural action of $Y \in \mathfrak{gl}_n$ on $\mathbb{C}^n$, i.e.

$$\Lambda^k Y\colon v_1 \wedge v_2 \wedge \ldots \wedge v_k \mapsto Yv_1 \wedge Yv_2 \wedge \ldots \wedge Yv_k,\ v_i \in \mathbb{C}^n.$$

We also set $c_0(Y) \equiv 1$.

Alternatively, one can define the elements c_k as the coefficients of the characteristic polynomial of Y:

$$\det(Y - z \cdot \mathrm{Id}_n) = \sum_{k=0}^{n} (-1)^{n-k} c_k(Y) \cdot z^{n-k}$$

Denote by y_{ij} the coordinates on $\mathfrak{gl}_n$ which correspond to standard matrix units $E_{ij} \in \mathfrak{gl}_n$. Then, $Y = [y_{ij}]_{i,j=1}^n$ and

$$c_k(Y) = \sum_{I \in \binom{[n]}{k}} \det Y_{II}. \tag{3.1}$$

It is well-known that elements $c_1, \ldots, c_n$ generate the ring of GL_n-invariants of $S(\mathfrak{gl}_n^*)$.

Proposition 3.1. *The ring $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ is a free polynomial ring in $c_1, \ldots, c_n$.*

3.2. **Construction of Kirillov algebra.** Let $\widetilde{\pi}\colon \mathrm{GL}_n \to V$ be a finite-dimensional representation of the group GL_n and let $\pi\colon \mathfrak{gl}_n \to \operatorname{End} V$ be the associated representation of Lie algebra $\mathfrak{gl}_n$. Define the so-called *Kirillov algebra*[1] of V as

$$\mathscr{C}(V) = (S(\mathfrak{gl}_n^*) \otimes \operatorname{End} V)^{\mathrm{GL}_n}.$$

In other words, Kirillov algebra is the algebra of GL_n-equivariant polynomial maps from $\mathfrak{gl}_n$ to $\operatorname{End} V$, i.e. for $F \in \mathscr{C}(V)$

$$F(\mathrm{Ad}(g)(Y)) = \widetilde{\pi}(g)F(Y)\widetilde{\pi}(g)^{-1},\ g \in \mathrm{GL}_n, Y \in \mathfrak{gl}_n, \tag{3.2}$$

Note that $\mathscr{C}(V)$ is an algebra over the ring $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$. The elements of $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$ are realized inside $\mathscr{C}(V)$ as scalar operators.

Then, we define the so-called *Kirillov operator* $\mathbf{D} = \mathbf{D}_V$ which acts on $\mathscr{C}(V)$ as follows: for any $F \in \mathscr{C}(V)$

$$(\mathbf{D}F)(Y) = \sum_{i,j=1}^{n} \frac{\partial F}{\partial y_{ji}}(Y) \cdot \pi(E_{ij}).$$

Remark 3.1. One can view Kirillov operator as a some kind of polarization operator.

It follows from the definition that for any positive integer p we have

$$(\mathbf{D}^p F)(Y) = \sum_{i_1,\ldots,i_p=1}^{n} \sum_{j_1,\ldots,j_p=1}^{n} \frac{\partial^p F}{\partial y_{j_1 i_1} \ldots \partial y_{j_p i_p}}(Y) \cdot \pi(E_{i_1 j_1} \ldots E_{i_p j_p}). \tag{3.3}$$

Alexander Kirillov in [6, Lemma 1] and [7, Section 1.4] hinted the following fact:

Proposition 3.2. *The operator $\mathbf{D}$ maps $\mathscr{C}(V)$ to itself.*

Proof. We first prove the following lemma.

Lemma 3.3. *For any $g \in \mathrm{GL}_n$ and any $1 \le k, l \le n$ we have the identity*

$$\sum_{i,j=1}^{n} [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \mathrm{Ad}(g)(E_{ij}) = E_{kl}.$$

Proof. Recall that on $\mathfrak{gl}_n$ there is a non-degenerate GL_n-invariant pairing, namely the *trace form* $(A, B) \mapsto \operatorname{tr}(AB)$. Denote the element of $\mathfrak{gl}_n$ on the left-hand side by A. The equality $A = E_{kl}$ is equivalent to $\mathrm{Ad}(g)^{-1}(A) = \mathrm{Ad}(g)^{-1}(E_{kl})$. In order to prove the latter, it suffices to verify that

$$\operatorname{tr}(\mathrm{Ad}(g)^{-1}(A)E_{pq}) = \operatorname{tr}(\mathrm{Ad}(g^{-1})(E_{kl})E_{pq}) \text{ for all } 1 \le p, q \le n.$$

Indeed, $\operatorname{tr}(\mathrm{Ad}(g^{-1})(E_{kl})E_{pq}) = \operatorname{tr}(E_{kl}\,\mathrm{Ad}(g)(E_{pq})) = [\mathrm{Ad}(g)(E_{pq})]_{kl}$. On the other hand,

$$\operatorname{tr}(\mathrm{Ad}(g)^{-1}(A)E_{pq}) = \sum_{i,j=1}^{n} [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \operatorname{tr}(E_{ij}E_{pq}) = \sum_{i,j=1}^{n} [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \delta_{iq}\delta_{jp} = [\mathrm{Ad}(g)(E_{pq})]_{lk},$$

which concluds the proof of the lemma. □

[1] Also known as *classical family algebra*, see Kirillov's original papers [6, 7].

Now let us return to the proof of Proposition 3.2. We know that for any $g \in \mathrm{GL}_n$ and $Y \in \mathfrak{gl}_n$ the identity $(F \circ \mathrm{Ad}(g^{-1}))(Y) = \widetilde{\pi}(g)F(Y)\widetilde{\pi}(g)^{-1}$ holds. Hence,

$$\frac{\partial}{\partial y_{ji}}\Big((F(\mathrm{Ad}(g)(Y)\Big) = \widetilde{\pi}(g) \cdot \frac{\partial F}{\partial y_{ji}}(Y) \cdot \widetilde{\pi}(g)^{-1}.$$

Then, we obtain

$$\widetilde{\pi}(g) \cdot (\mathbf{D}F)(Y) \cdot \widetilde{\pi}(g)^{-1} = \sum_{i,j=1}^{n} \widetilde{\pi}(g) \frac{\partial F}{\partial y_{ji}}(Y) \cdot \pi(E_{ij})\widetilde{\pi}(g)^{-1} = \sum_{i,j=1}^{n} \widetilde{\pi}(g) \frac{\partial F}{\partial y_{ji}}(Y)\widetilde{\pi}(g)^{-1} \cdot \pi(\mathrm{Ad}(g)(E_{ij})) =$$

$$= \sum_{i,j=1}^{n} \frac{\partial}{\partial y_{ji}}\Big((F(\mathrm{Ad}(g)(Y)\Big) \cdot \pi(\mathrm{Ad}(g)(E_{ij})) = \sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \frac{\partial F}{\partial y_{lk}}(\mathrm{Ad}(g)(Y)) \cdot [\mathrm{Ad}(g)(E_{ji})]_{lk} \cdot \pi(\mathrm{Ad}(g)(E_{ij})) =$$

$$= \sum_{k,l=1}^{n} \frac{\partial F}{\partial y_{lk}}(\mathrm{Ad}(g)(Y)) \cdot \pi(E_{kl}) = (\mathbf{D}F)(\mathrm{Ad}(g)(Y)),$$

due to lemma above. Thus, $\mathbf{D}F$ satsifies (3.2) and hence, belongs to $\mathscr{C}(V)$. ☐

Remark 3.2. One can generalize the construction of Kirillov algebra, operator $\mathbf{D}$ and the results of this subsection for any semi-simple Lie algebra $\mathfrak{g}$. The corresponding operator $\mathbf{D}$ can be described using a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g. the Killing form in the case of simple $\mathfrak{g}$).

The explicit formulas for specific elements of $\mathscr{C}(V)$ are often quite complicated. However, these formulas can be simplified if one restrict elements of Kirillov algebra to Cartan subalgebra $\mathfrak{h}$. Moreover, these restrictions uniquely determine the elements of $\mathscr{C}(V)$:

Proposition 3.4. *The restriction map* $F \mapsto F|_{\mathfrak{h}}$, $F \in \mathscr{C}(V)$ *is an injective algebra homomorphism.*

Proof. We use the argument outlined in [6, Theorem 2]. It follows from the equality (3.2) that the restriction $F|_{\mathfrak{h}}$ completely determines the map $F\colon \mathfrak{gl}_n \to \operatorname{End}\mathcal{P}(n,r)$ on the set of all regular semi-simple elements of $\mathfrak{gl}_n$ (those are conjugate to elements of $\mathfrak{h}$). Since the latter set is Zariski dense in $\mathfrak{gl}_n$ the claim follows. ☐

The following statement shows that in order to study the Kirillov algebra of V one can study the Kirillov algebra of a "larger" representation.

Proposition 3.5. *Let* $V = \bigoplus_\alpha V_\alpha$ *be a direct sum of* GL_n*-modules. Then, the natural algebra homomorphism* $\bigotimes_\alpha \mathscr{C}(V_\alpha) \to \mathscr{C}(V)$ *induced by the embeddings* $\iota_\alpha\colon \operatorname{End}(V_\alpha) \hookrightarrow \operatorname{End}(V)$ *is injective.*

Proof. The map in question sends a family of maps $F_\alpha\colon \mathfrak{gl}_n \to \operatorname{End} V_\alpha$ to a map $\sum_\alpha \iota_\alpha \circ F_\alpha\colon \mathfrak{gl}_n \to \mathscr{C}(V)$. Note that this expression is well defined since for every $Y \in \mathfrak{gl}_n$ the summand $(\iota_\alpha \circ F_\alpha)(Y)$ belongs to $\iota_\alpha(\operatorname{End} V_\alpha)$. In particular, this map is indeed an algebra homomorphism. The injectivity now follows from the injectivity of maps ι_α. ☐

3.3. **Kirillov algebra for $\mathcal{P}(n,r)$.** Now consider the $\mathfrak{gl}_n$-module $\mathcal{P}(n,r) = \mathbb{C}[\mathrm{Mat}(n,r)]$. By [14, Ch. VII, §49] (see also [3, Proposition 3]) the highest weight vectors of $\mathcal{P}(n,r)$ are as follows:

$$\begin{vmatrix} x_{i_1 1} \end{vmatrix}^{p_1} \cdot \begin{vmatrix} x_{i_1 1} & x_{i_1 2} \\ x_{i_2 1} & x_{i_2 2} \end{vmatrix}^{p_2} \cdot \ldots \cdot \begin{vmatrix} x_{i_1 1} & \ldots & x_{i_1 l} \\ \vdots & \ddots & \vdots \\ x_{i_l 1} & \ldots & x_{i_l l} \end{vmatrix}^{p_l},$$

where $0 \le l \le \min\{r,n\}$, $p_1,\ldots,p_l$ are arbitrary non-negative integers and $i_1,\ldots,i_l$ are arbitrary distinct elements of $\{1,\ldots,r\}$. The element above is of weight $(p_1+\ldots+p_l,\ldots,p_{l-1}+p_l,p_l,0,\ldots,0)$.

On matrix space $\mathrm{Mat}(n,r)$ there are natural actions of groups GL_n and GL_r which induce actions of these groups on $\mathcal{P}(n,r)$. Namely, for any $A \in \mathrm{Mat}(n,r)$, $g \in \mathrm{GL}_n$ and $h \in \mathrm{GL}_r$ we have

$$\widetilde{L}(g)(P)(A) = P(g^T \cdot A),\ \widetilde{R}(h)(P)(A) = P(A \cdot h).$$

Note that these two actions commute and give rise to an action of $\mathrm{GL}_n \times \mathrm{GL}_r$ on $\mathcal{P}(n,r)$. Then, *Howe duality* (see [4, Section 2.1.2]) implies that the $\mathrm{GL}_n \times \mathrm{GL}_r$-module $\mathcal{P}(n,r) = \mathbb{C}[\mathrm{Mat}(n,r)]$ decomposes as follows:

$$\mathcal{P}(n,r) \simeq \bigoplus_{\ell(\lambda)\le r} V(\lambda) \otimes W(\lambda), \tag{3.4}$$

where the summation is over all partitions $\lambda = (\lambda_1,\ldots,\lambda_n)$ of length at most r (i.e. with $\lambda_i = 0$ for $i > r$). Here $V(\lambda)$ and $W(\lambda)$ denote the irreducible representations of the highest weight λ of GL_n and GL_r, respectively. In particular, we conclude that $\mathcal{P}(n,r)$ contains as subrepresentations all irreducible representations of $\mathfrak{gl}_n$ that have highest weight λ with $\ell(\lambda) \le r$.

Let us now apply the Kirillov algebra construction for GL_n-module $\mathcal{P}(n,r)$. In view of Proposition 3.5 and decomposition (3.4), the algebra $\mathscr{C}(\mathcal{P}(n,r))$ contains $\mathscr{C}(V(\lambda))$ as a subalgebra for all dominant weights $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\lambda_1 \ge \ldots \lambda_n \ge 0$ and $\ell(\lambda) \le r$.

3.4. **Big and medium algebras.** Recall that big algebra $\mathscr{B}(V)$ in type A is the subalgebra of $\mathscr{C}(V)$ generated by elements

$$M_{p,q} = \mathbf{D}^q(c_{p+q}),\ 0 \le p, q \le n,\ p+q \le n.$$

The medium algebra $\mathscr{M}(V)$ of $\mathcal{P}(n,r)$ is a subalgebra of $\mathscr{B}(V)$ defined as follows:

$$\mathscr{M}(V) = \langle F, \mathbf{D}(F) : F \in S(\mathfrak{gl}_n^*)^{\mathrm{GL}_n}\rangle \subset \mathscr{B}(V).$$

Now let us give a more explicit formula for the generators of $\mathscr{B}(V)$. Recall that $\pi\colon \mathfrak{gl}_n \to \operatorname{End} V$ is the representation of $\mathfrak{gl}_n$ which comes from GL_n-action on V.

Proposition 3.6. *For any $Y \in \mathfrak{gl}_n$ we have*

$$M_{p,q}(Y) = \mathbf{D}^q(c_{p+q})(Y) = \sum_{\substack{I_1,J_1 \in \binom{[n]}{p} \\ I_2,J_2\in\binom{[n]}{q} \\ I_1\sqcup I_1 = J_1 \sqcup J_2}} \operatorname{sgn}\begin{pmatrix} I_1 & I_2 \\ J_1 & J_2\end{pmatrix} \det Y_{I_1J_1} \cdot \operatorname{symdet} \pi(E)_{J_2I_2}. \tag{3.5}$$

Proof. We have

$$c_{p+q}(Y) = \sum_{I\in\binom{[n]}{p+q}} \sum_{\sigma\in\mathfrak{S}_{p+q}} \operatorname{sgn}(\sigma) \prod_{s\in[p+q]} y_{i_s i_{\sigma(s)}} = \sum_{I\in\binom{[n]}{p+q}} \operatorname{sgn}(\sigma) y_{i_1 i_{\sigma(1)}} \dots y_{i_{p+q} i_{\sigma(p+q)}},$$

where the first summation runs over all $(p+q)$-element subsets $I = \{i_1, \dots, i_{p+q}\}$ of $[n]$. Here we identify $\binom{[n]}{p+q}$ with $\mathfrak{S}_{p+q}\backslash[n]^{\underline{p+q}}$ (see Remark 2.1). Applying the operator $\mathbf{D}^q$ (see also (3.3)) we obtain

$$\begin{aligned}
\mathbf{D}^q(c_{p+q})(Y) &= \sum_{I\in\binom{[n]}{p+q}} \sum_{\sigma\in\mathfrak{S}_{p+q}} \operatorname{sgn}(\sigma) \sum_{V\in\binom{[p+q]}{q}} \prod_{v\in V^c} y_{i_v i_{\sigma(v)}} \sum_{\tau\in\mathfrak{S}(V)} \prod_{v\in V} \pi(E_{i_{\sigma(\tau(v))} i_{\tau(v)}}) = \\
&= \sum_{I\in\binom{[n]}{p+q}} \sum_{V\in\binom{[p+q]}{q}} \sum_{[\sigma]\in\mathfrak{S}_{p+q}/\mathfrak{S}(V^c)} \sum_{\tau_1\in\mathfrak{S}(V^c)} \operatorname{sgn}(\sigma\tau_1) \prod_{v\in V^c} y_{i_v i_{\sigma(\tau_1(v))}} \sum_{\tau_2\in\mathfrak{S}(V)} \prod_{v\in V} \pi(E_{i_{\sigma(\tau_2(v))} i_{\tau_2(v)}}) = \\
&= \sum_{I\in\binom{[n]}{p+q}} \sum_{V\in\binom{[p+q]}{q}} \sum_{[\sigma]\in\mathfrak{S}_{p+q}/\mathfrak{S}(V^c)} \det(Y_{I(V^c),(\sigma^{-1}I)(V^c)}) \operatorname{sgn}(\sigma) \sum_{\tau_2\in\mathfrak{S}(V)} \prod_{v\in V} \pi(E_{i_{\sigma(\tau_2(v))} i_{\tau_2(v)}}) = \\
&= \sum_{I\in\binom{[n]}{p+q}} \sum_{V\in\binom{[p+q]}{q}} \sum_{[\sigma]\in\mathfrak{S}_{p+q}/\mathfrak{S}(V)\times\mathfrak{S}(V^c)} \operatorname{sgn}(\sigma) \det(Y_{I(V^c),(\sigma^{-1}I)(V^c)}) \operatorname{symdet} \pi(E)_{(\sigma^{-1}I)(V),I(V)},
\end{aligned}$$

where we denoted $V^c = [p+q] \setminus V$. To conclude the proof it remains to note that the summation over $I \in \binom{[n]}{p+q}$, $V \in \binom{[p+q]}{p}$ and $[\sigma] \in \mathfrak{S}_{p+q}/\mathfrak{S}(V) \times \mathfrak{S}(V^c)$ is equivalent to the summation over I_1, I_2, J_1 and J_2 as in (3.5). Indeed, if we put

$$I_1 = I(V^c),\ J_1 = (\sigma^{-1}I)(V^c),\ I_2 = I(V),\ J_2 = (\sigma^{-1}I)(V),$$

we would get the right-hand side of (3.5) since $\operatorname{sgn}(\sigma) = \operatorname{sgn}\binom{I_1\ I_2}{J_1\ J_2}$. □

The next proposition relates big (resp. medium) algebras of direct sums with big (resp. medium) algebras of summands.

Proposition 3.7. *(i) The image of the algebra homomorphism $\bigotimes_\alpha \mathscr{C}(V_\alpha) \to \mathscr{C}(V)$ from Proposition 3.5 contains the big algebra $\mathscr{B}(V)$.*
(ii) For each α there exist surjective homomorphisms $\mathscr{B}(V) \to \mathscr{B}(V_\alpha)$ and $\mathscr{M}(V) \to \mathscr{M}(V_\alpha)$.

Proof. The definition of the Kirillov operator implies that subalgebra $\bigotimes_\alpha \mathscr{C}(V_\alpha)$ inside $\mathscr{C}(V)$ is stable under the operator $\mathbf{D}_V$. Moreover, the restriction of $\mathbf{D}_V$ to $\bigotimes_\alpha \mathscr{C}(V_\alpha)$ coincides with $\bigotimes_\alpha \mathbf{D}_{V_\alpha}$. The first part now follows from the fact that $c_1, \dots, c_n \in S(\mathfrak{gl}_n^*)^{\mathrm{GL}_n}$ are realized inside $\mathscr{C}(V)$ as scalar operators and the identity $\mathbf{D}_V^p(c_k \otimes \mathrm{id}_V) = \bigotimes_\alpha \mathbf{D}_{V_\alpha}^p(c_k \otimes \mathrm{id}_{V_\alpha})$ for all $p \ge 1$. The required homomorphisms $\mathscr{B}(V) \to \mathscr{B}(V_\alpha)$ and $\mathscr{M}(V) \to \mathscr{M}(V_\alpha)$ send $\mathbf{D}_V^p(c_k \otimes \mathrm{id}_V)$ to $\mathbf{D}_{V_\alpha}^p(c_k \otimes \mathrm{id}_{V_\alpha})$. □

3.4.1. *Generators of big algebra.* From now on we consider the case $V = \mathcal{P}(n,r)$. Proposition 3.7 implies that many properties of big algebras of irreducible representations can be read off from $\mathscr{B}(\mathcal{P}(n,r))$.

Observe that the big algebra $\mathscr{B}(\mathcal{P}(n,r))$ is contained in $S(\mathfrak{gl}_n^*) \otimes \mathcal{PD}(n,r)$. In other words, elements of big algebra are certain differential operators on $\mathcal{P}(n,r)$ whose coefficients are polynomials in variables $x_{i\alpha}$ and y_{jk}. One of the main results of this note is the explicit formula (the so-called *normal form*) for operators $M_{p,q}$.

Theorem 3.8. *The normal form of the operator $M_{p,q}(Y)$ is as follows:*

$$M_{p,q}(Y) = \sum_{\ell=0}^{q} (-1)^{q-\ell}(q-\ell)!\,\ell!\left\{\begin{matrix} q \\ \ell \end{matrix}\right\}\times \tag{3.6}$$

$$\times \sum_{\substack{I_1,J_1\in\binom{[n]}{p}\\ I_2,J_2\in\binom{[n]}{q}\\ I_1\sqcup I_1=J_1\sqcup J_2}} \operatorname{sgn}\begin{pmatrix} I_1 & I_2 \\ J_1 & J_2\end{pmatrix}\det Y_{I_1J_1}\sum_{R\in\binom{[r]}{\ell}}\sum_{V,W\in\binom{[q]}{\ell}} \varepsilon(J_2,I_2,J_2(V),I_2(W))\det(X_{J_2(V),R})\det(D_{I_2(W),R}).$$

Corollary 3.9. *Big algebra $\mathscr{B}(\mathcal{P}(n,r))$ is generated by operators $\{F_{p,q} : p,q\ge 0, p+q\le n\}$, where*

$$F_{p,q}(Y) = \sum_{\substack{I_1,J_1\in\binom{[n]}{p}\\ I_2,J_2\in\binom{[n]}{q}\\ I_1\sqcup I_2=J_1\sqcup J_2}} \operatorname{sgn}\begin{pmatrix} I_1 & I_2 \\ J_1 & J_2\end{pmatrix}\det Y_{I_1J_1}\sum_{R\in\binom{[r]}{q}}\det(X_{J_2,R})\det(D_{I_2,R}) \tag{3.7}$$

In particular, operators $\{M_{p,q} : p,q\ge 0, p+q\le n\}$ and $\{F_{p,q} : p,q\ge 0, p+q\le n\}$ are related to each other as follows:

$$M_{p,q} = \sum_{\ell=0}^{q}(-1)^{q-\ell}(q-\ell)!\,\ell!\left\{\begin{matrix} q \\ \ell \end{matrix}\right\}\binom{n-p-\ell}{q-\ell}F_{p,\ell}. \tag{3.8}$$

We prove these formulas for $M_{p,q}$ and $F_{p,q}$ in the next section.

Remark 3.3. Note that both sets $\{F_{p,0}\}_{p=1}^n$ and $\{M_{p,0}\}_{p=1}^n$ are generator sets for $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$.

3.4.2. *Restriction to Cartan subalgebra.* The formulas (3.6) and (3.7) become simpler if we restrict $Y\in\mathfrak{gl}_n$ to Cartan subalgebra $\mathfrak{h}$, i.e. to diagonal matrices.

Proposition 3.10. *For $Y = \operatorname{diag}(z_1,\dots,z_n)\in\mathfrak{h}$ we have*

$$F_{p,q}(Y) = \sum_{\substack{I\in\binom{[n]}{p}\\ J\in\binom{[n]}{q}\\ I\cap J=\varnothing}}\prod_{i\in I} z_i\sum_{R\in\binom{[r]}{q}}\det(X_{JR})\det(D_{JR}).$$

Proof. This is a consequence of Corollary 3.9 and the observation that for $Y\in\mathfrak{h}$ the determinant $\det Y_{I_1J_1}$ vanishes unless $I_1 = J_1$. □

We use this formula in the proof of commutativity of the big algebra (Theorem 7.1).

4. Proofs of Theorem 3.8 and Corollary 3.9

In this section we prove Theorem 3.8 and Corollary 3.9, i.e. formulas (3.6) and (3.7). The proof is purely computational and reduces to some identities in Weyl algebra.

4.1. **Computation of the symmetrized determinants.** In view of Proposition 3.6, to prove Theorem 3.8 we first need to get a formula for symdet $L(E)_{IJ}$. Recall that for k-tuples $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_k)$ in $[n]^{\underline{k}}$ the symmetrized determinant symdet $L(E_{IJ})$ is defined as

$$\operatorname{symdet} L(E_{IJ}) = \sum_{\sigma,\tau\in\mathfrak{S}_k}\operatorname{sgn}(\sigma\tau)L(E_{i_{\sigma(1)}i_{\tau(1)}})\dots L(E_{i_{\sigma(k)}i_{\tau(k)}}) =$$

$$= \sum_{a_1,\dots,a_k=1}^{n}\sum_{\sigma,\tau\in\mathfrak{S}_k}\operatorname{sgn}(\sigma\tau)\cdot x_{i_{\sigma(1)}a_1}\partial_{j_{\tau(1)}a_1}\dots x_{i_{\sigma(k)}a_k}\partial_{j_{\tau(k)}a_k}.$$

Our aim is to obtain the reduced expression (the normal form) for symdet $L(E_{IJ})$ in the algebra $\mathcal{PD}(n,r)$ of differential operators on $\mathfrak{gl}_n$ with polynomial coefficients. The computation is divided into several steps.

We start with the following auxuliary identity.

Lemma 4.1. *Let N be a positive integer. For any permutations $\sigma,\tau\in\mathfrak{S}_N$ and any $s\in\{1,\dots,N\}$ define*

$$\delta_s(\sigma,\tau) = \begin{cases} 0, & \text{if } \sigma(s)\le\tau(s),\\ 1, & \text{if } \sigma(s)>\tau(s).\end{cases}$$

Then, for any $\alpha_1,\dots,\alpha_N$ the following identity holds:

$$\sum_{\sigma,\tau\in\mathfrak{S}_N}\operatorname{sgn}(\sigma\tau)\cdot(\alpha_1+\delta_1(\sigma,\tau))\dots(\alpha_N+\delta_N(\sigma,\tau)) = (-1)^{N-1}(N-1)!(\alpha_1+\dots+\alpha_N). \tag{4.1}$$

Proof. The identity clearly holds for $N = 1$, so let us assume that $N \geq 2$. Observe that the left-hand side can be expressed as

$$\sum_{\ell=0}^{N} C_\ell \cdot e_\ell(\alpha_1, \ldots, \alpha_N),$$

where $e_\ell(\alpha_1, \ldots, \alpha_N)$ is the ℓ-th elementary symmetric polynomial $\alpha_1, \ldots, \alpha_N$ and $C_0, \ldots, C_\ell$ are certain real numbers. Therefore, to check the identity (4.1) it sufices to calculate coefficients C_ℓ for every $0 \leq \ell \leq N$. Note that C_ℓ is equal to the coefficient of the term $\alpha_1 \ldots \alpha_\ell$ of the left-hand side. Then, it is not difiicult to see that

$$C_\ell = \sum_{\sigma,\tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau) \prod_{s=\ell+1}^{N} \delta_s(\sigma, \tau).$$

In other words, C_ℓ is the sum of $\operatorname{sgn}(\sigma\tau)$, where (σ, τ) runs over all pairs of permutations in $\mathfrak{S}_N$ such that

$$\sigma(s) > \tau(s) \text{ for all } \ell + 1 \leq s \leq N.$$

Let $\Gamma_\ell \subset \mathfrak{S}_N \times \mathfrak{S}_N$ be the set of all such pairs. Now let us consider several cases:

- $\ell = 0$. In this case we have
$$C_0 = \sum_{\sigma,\tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau) \cdot \delta_1(\sigma, \tau) \ldots \delta_N(\sigma, \tau) = 0$$
because for any $\sigma, \tau \in \mathfrak{S}_N$ at least one of $\delta_s(\sigma, \tau)$ is zero (for example one can take $s = \sigma^{-1}(1)$).
- $\ell = 1$. We claim that in this case the set $\Gamma_\ell = \Gamma_{N-1}$ contains exactly $(N-1)!$ elements. Indeed, by definition a pair $(\sigma, \tau) \in \mathfrak{S}_N \times \mathfrak{S}_N$ belongs to Γ_{N-1} if $\sigma(s) > \tau(s)$ for every $s \in \{2, \ldots, N\}$. One checks that this holds if and only if these permutations satisfy $\sigma(1) = 1$, $\tau(1) = N$ and $\sigma(s) = \tau(s) + 1$ for all $s = 2, \ldots, N$. In particular, $|\Gamma_{N-1}| = (N-1)!$ and for any $(\sigma, \tau) \in \Gamma_{N-1}$ the permutation $\sigma\tau^{-1}$ is the cycle $(1\ 2\ \ldots\ N)$. Hence,
$$C_1 = \sum_{(\sigma,\tau) \in \Gamma_{N-1}} \operatorname{sgn}(\sigma\tau) = |\Gamma_{N-1}| \cdot (-1)^{N-1} = (-1)^{N-1} \cdot (N-1)!.$$
- $2 \leq \ell \leq N$. Note that for any permutations $\sigma', \tau' \in \mathfrak{S}_N$ which fix each of $\ell + 1, \ldots, N$ the pair (σ, τ) belongs to $\Gamma_\ell = \Gamma_N$ if and only if $(\sigma\sigma', \tau\tau') \in \Gamma_N$. This and the equality
$$\sum_{\sigma',\tau' \in \mathfrak{S}_\ell} \operatorname{sgn}(\sigma'\tau') = \left(\sum_{\sigma' \in \mathfrak{S}_\ell} \operatorname{sgn} \sigma'\right)\left(\sum_{\tau' \in \mathfrak{S}_\ell} \operatorname{sgn} \tau'\right) = 0$$
imply that $C_\ell = 0$ for all $2 \leq \ell \leq N$.

Combining everything together, we obtain

$$C_\ell = \begin{cases} 0, & \ell \neq 1, \\ (-1)^{N-1}(N-1)!, & \ell = 1, \end{cases}$$

which is equivalent to (4.1). □

Corollary 4.2. *Let N be a positive integer and let $\ell \in \{0, 1, \ldots, N\}$. Then, in the notation of Lemma 4.1 we have*

$$\sum_{\sigma,\tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau)(\alpha_1 + \delta_1(\sigma, \tau)) \ldots (\alpha_\ell + \delta_\ell(\sigma, \tau)) = \begin{cases} 0, & \ell \in \{0, 1, \ldots, N-2\}, \\ (-1)^{N-1}(N-1)!, & \ell = N-1, \\ (-1)^{N-1}(N-1)! \cdot (\alpha_1 + \ldots + \alpha_N), & \ell = N. \end{cases}$$

Remark 4.1. For $\ell = 0$ we define the left-hand side as $\sum_{\sigma,\tau \in \mathfrak{S}_N} \operatorname{sgn}(\sigma\tau)$.

Proof. This follows from (4.1) after taking derivatives of both sides with respect to $\alpha_{\ell+1}, \ldots, \alpha_N$. □

Next we prove the following identity in Weyl algebra which essentially computes the symmetrized determinant $\operatorname{symdet} L(E)_{IJ}$ in the simplest case $r = 1$.

Lemma 4.3. *Consider the Weyl algebra generated by variables $u_1, \ldots, u_n$ and the corresponding partial derivatives $\partial_1, \ldots, \partial_n$. Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ be two k-tuples in $[n]^{\underline{k}}$. Assume that I and J have ℓ common elements, $0 \leq \ell \leq k$. Denote*

$$\Psi(I, J) = \operatorname{symdet}\left([u_{i_\alpha}\partial_{j_\beta}]_{\alpha,\beta=1}^{k}\right) = \sum_{\sigma,\tau \in \mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) u_{i_{\sigma(1)}}\partial_{j_{\tau(1)}} \ldots u_{i_{\sigma(k)}}\partial_{j_{\tau(k)}}. \quad (4.2)$$

Then, $\Psi(I, J)$, as an element of Weyl algebra, can be simplified as follows:

- *if $\ell = k$ and $\pi \in \mathfrak{S}_k$ is such that $j_l = i_{\pi(l)}$ for all $l \in \{1, \ldots, k\}$, then $\Psi(I, J) = (-1)^{k-1}(k-1)! \cdot \operatorname{sgn}(\pi) \sum_{i \in I} u_i\partial_i$;*

- *if* $\ell = k-1$ *and* $\pi \in \mathfrak{S}_k$ *is such that* $j_l = i_{\pi(l)}$ *for all* $l \in \{1, \dots, k-1\}$ *and* $j_k \neq i_{\pi(k)}$*, then* $\Psi(I,J) = (-1)^{k-1}(k-1)! \cdot \operatorname{sgn}(\pi) u_{i_{\pi(k)}} \partial_{j_k}$*;*
- *if* $\ell \le k-2$*, then* $\Psi(I,J) = 0$*.*

Moreover, viewing I *and* J *as functions on* $[k] = \{1, \dots, k\}$ *one can express* $\Psi(I,J)$ *as follows:*

$$\Psi(I,J) = (-1)^{k-1}(k-1)! \sum_{\pi \in \mathfrak{S}_k} \sum_{s=1}^{k} \mathbb{1}(I|_{[k]\setminus\{s\}} = J|_{[k]\setminus\{s\}}) \cdot \operatorname{sgn}(\pi) u_{i_s} \partial_{j_{\pi(s)}}.$$

Proof. We start with the following observation: for any permutation $\pi \in \mathfrak{S}_k$ we have $\Psi(\pi \cdot I, J) = \operatorname{sgn}(\pi) \cdot \Psi(I,J)$, where $\pi \cdot I = (i_{\pi^{-1}(1)}, \dots, i_{\pi^{-1}(k)})$. Therefore, it suffices to prove the statement in the case when $i_1 = j_1, \dots, i_\ell = j_\ell$ and

$$\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_k\} = \{i_1, \dots, i_\ell\}.$$

Recall that $\ell = |\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_k\}|$.

Denote $K = (i_1, \dots, i_\ell) = (j_1, \dots, j_\ell)$. To simplify the notation let us also assume that $K = (1, \dots, \ell)$. Now note that elements $u_{i_{\ell+1}}, \dots, u_{i_k}$ and $\partial_{j_{\ell+1}}, \dots, \partial_{j_k}$ commute with each other and also with $\{u_i, \partial_i\}_{i \in K}$. Therefore, we can rewrite $\Psi(I,J)$ as

$$\Psi(I,J) = u_{i_{\ell+1}} \dots u_{i_k} \partial_{j_{\ell+1}} \dots \partial_{j_k} \cdot \Phi(\ell; K), \tag{4.3}$$

where $\Phi(\ell; K)$ is a certain element in Weyl algebra generated by u_i, ∂_i with $i \in K$. Namely, $\Phi(\ell; K)$ is obtained from the expression for $\Phi(I,J)$ by removing all u_i and ∂_i with $i \notin K$. One finds that the action of $\Phi(\ell; K)$ on a monomial $u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}$, where $\alpha_1, \dots, \alpha_\ell \in \mathbb{N}_0$, is given by

$$\Phi(\ell;K)(u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}) = \left(\sum_{\sigma,\tau \in \mathfrak{S}_k} \operatorname{sgn}(\sigma\tau)(\alpha_1 + \delta_1(\sigma^{-1}, \tau^{-1})) \dots (\alpha_\ell + \delta_\ell(\sigma^{-1}, \tau^{-1})) \right) \cdot u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell},$$

where δ_i's are defined as in Lemma 4.1. Indeed, this follows from the identities

$$(u_i\partial_i)(u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}) = \alpha_i \cdot u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}, \ (\partial_i u_i)(u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}) = (\alpha_i + 1) \cdot u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}, \ 1 \le i \le \ell.$$

Now observe that

$$\sum_{\sigma,\tau \in \mathfrak{S}_k} \operatorname{sgn}(\sigma\tau)(\alpha_1 + \delta_1(\sigma^{-1}, \tau^{-1})) \dots (\alpha_\ell + \delta_\ell(\sigma^{-1}, \tau^{-1})) = \sum_{\sigma,\tau \in \mathfrak{S}_k} (\alpha_1 + \delta_1(\sigma,\tau)) \dots (\alpha_\ell + \delta_\ell(\sigma,\tau)),$$

and hence, Corollary 4.2 implies that

- if $0 \le \ell \le k-2$, then $\Phi(\ell;K)(u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}) = 0$;
- if $\ell = k-1$, then $\Phi(\ell;K)(u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}) = (-1)^{k-1}(k-1)! \cdot u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}$,
- if $\ell = k$, then $\Phi(\ell;K)(u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}) = (-1)^{k-1}(k-1)! \cdot (\alpha_1 + \dots + \alpha_k) u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}$.

Since the elements of Weyl algebra are uniquely defined by their action on the corresponding polynomial ring $\mathbb{C}[u_1, \dots, u_n]$ we obtain

$$\Phi(\ell;K) = \begin{cases} 0, & 0 \le \ell \le k-2, \\ (-1)^{k-1}(k-1)!, & \ell = k-1, \\ (-1)^{k-1}(k-1)! \sum_{i \in K} u_i \partial_i, & \ell = k. \end{cases}$$

Combining this with (4.3) concludes the proof. □

Lemma 4.4. *Fix* $a_1, \dots, a_k \in \{1, \dots, r\}$*. For any two* k*-tuples* $I = (i_1, \dots, i_k)$ *and* $J = (j_1, \dots, j_k)$ *of distinct elements of* $\{1, \dots, n\}$ *denote*

$$\Delta(I,J) = \sum_{\sigma,\tau \in \mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) \cdot x_{i_{\sigma(1)}a_1} \partial_{j_{\tau(1)}a_1} \dots x_{i_{\sigma(k)}a_k} \partial_{j_{\tau(k)}a_k}.$$

Then, we have the following identity

$$\Delta(I,J) = (-1)^{k-\ell}(k-\ell)! \sum_{V,W \in \binom{[k]}{\ell}} \varepsilon(I,J,I(V),J(W)) \det(X_{I(V),R}) \cdot \det(D_{J(W),R}), \tag{4.4}$$

where $R = \{a_1, \dots, a_k\}$ *and* $\ell = |R|$*.*

Remark 4.2. Here $I(V)$ and $J(W)$ are the subtuples of I and J that correspond to V and W, respectively. The k-element subsets V and W are viewed as k-tuples with the ordering chosen in an arbitrary way (see Remark 2.1). Note that the summand does not depend on a choice of the ordering of V and W.

Remark 4.3. Observe that the formula for $\Psi(I,J)$ from Lemma 4.3 is a particular case of (4.4) when $\ell = |R| = 1$. Indeed, for any $I, J \in [n]^{\underline{k}}$

$$\sum_{\pi\in\mathfrak{S}_k}\sum_{s=1}^{k} \mathbb{1}(I|_{[k]\setminus\{s\}} = J|_{[k]\setminus\{s\}})\cdot \operatorname{sgn}(\pi) = \varepsilon(I,J,s,s),$$

where in $\varepsilon(I,J,s,s)$ the element s is viewed as a 1-tuple.

Remark 4.4. In particular, when $I = J$, the formula (4.4) can be simplified as follows:

$$\begin{aligned}\Delta(I,I) &= (-1)^{k-\ell}(k-\ell)! \sum_{V\in\binom{[k]}{\ell}} \det([x_{I(v),l}]_{v\in V, l\in R})\cdot\det([\partial_{I(v),l}]_{v\in V,l\in R}) =\\ &= (-1)^{k-\ell}(k-\ell)!\sum_{V\in\binom{[k]}{\ell}} \det(X_{I(V),R})\cdot\det(D_{I(V),R}).\end{aligned}$$

Proof. Define the decomposition $\{1,\dots,k\} = \bigsqcup_{l=1}^{r} K_l$ via

$$K_l = \{j\in\{1,\dots,k\} : a_j = l\}.$$

Let m_l be the cardinality of K_l for each l. Note that $R = \{l\in[r] : m_l > 0\}$ and $\ell = |R|$. Denote by $\mathfrak{S}(K_l)$ the group of permutations of the set K_l viewed as a subgroup of $\mathfrak{S}_k$. In particular, $\mathfrak{S}(K_l)\simeq\mathfrak{S}_{m_l}$ for all l. We also denote by $\mathfrak{S}(K)$ the subgroup of $\mathfrak{S}_k$ which stabilizes each of the subsets R_l, i.e. $\mathfrak{S}(K) = \mathfrak{S}(K_1)\times\ldots\times\mathfrak{S}(K_r)$.

Towards the end of the proof we will regard any sequence $\{c_j\}_{j\in K_l}$ indexed by elements of K_l as an m_l-tuple by considering the elements of K_l in ascending order. Since we are working in a non-commutative algebra let us make a convention that products over K_l are considered in the ascending ordering as well.

For any permutation $\sigma\in\mathfrak{S}_k$ and any $l\in[r]$ define the m_l-tuples $\sigma^{-1}I|_{K_l}$ and $\sigma^{-1}J|_{K_l}$ as follows (cf. (2.1)):

$$\sigma^{-1}I|_{K_l} = \{i_{\sigma(s)}\}_{s\in K_l},\ \sigma^{-1}I|_{K_l} = \{j_{\sigma(s)}\}_{s\in K_l}.$$

For any two p-tuples $U = (u_1,\dots,u_p)$ and $V = (v_1,\dots,v_p)$ in $[k]^{\underline{p}}$ define

$$\Theta_l(U,V) = \prod_{i=1}^{p} x_{u_i l}\partial_{v_i l} = x_{u_1 l}\partial_{v_1 l}\dots x_{u_p l}\partial_{v_p l}.$$

We also denote (cf. Lemma 4.3)

$$\Psi_l(U,V) = \begin{cases}1, & l\notin R,\\ \operatorname{symdet}\left([x_{u_\alpha l}\partial_{v_\beta l}]_{\alpha,\beta=1}^{p}\right), & l\in R.\end{cases}$$

Observe that for any $\pi_1,\pi_2\in\mathfrak{S}(K)$ we have

$$\Psi_l((\sigma\pi_1)^{-1}I|_{K_l},(\tau\pi_2)^{-1}J|_{K_l}) = \operatorname{sgn}(\pi_1|_{K_l})\operatorname{sgn}(\pi_2|_{K_l})\cdot\Psi(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l}), \tag{4.5}$$

where $\pi_i|_{K_l}$ is the restriction of permutation $\pi_i\in\mathfrak{S}(K)$ to subset K_l (recall that $\pi_i(K_l) = K_l$). Finally, denote by $\mathfrak{S}_k/\mathfrak{S}(K)$ the set of all left cosets of $\mathfrak{S}(K)$ in $\mathfrak{S}_k$.

With all these notations we can now proceed to the proof of (4.4). Since x_{ia} and ∂_{jb} commute whenever $a\neq b$, we can rewrite $\Delta(I,J)$ as follows:

$$\begin{aligned}\Delta(I,J) &= \sum_{\sigma,\tau\in\mathfrak{S}_k}\operatorname{sgn}(\sigma\tau)\cdot x_{i_{\sigma(1)}a_1}\partial_{j_{\tau(1)}a_1}\dots x_{i_{\sigma(k)}a_k}\partial_{j_{\tau(k)}a_k} =\\ &= \sum_{\sigma,\tau\in\mathfrak{S}_k}\operatorname{sgn}(\sigma\tau)\cdot\prod_{l=1}^{r}\Theta_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l}) =\\ &= \sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)}\ \sum_{\pi_1,\pi_2\in\mathfrak{S}(K)}\operatorname{sgn}(\sigma\pi_1\tau\pi_2)\cdot\prod_{l=1}^{r}\Theta_l((\sigma\pi_1)^{-1}I|_{K_l},(\tau\pi_2)^{-1}J|_{K_l}) =\\ &= \sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)}\operatorname{sgn}(\sigma\tau)\left(\sum_{\pi_1,\pi_2\in\mathfrak{S}(K)}\operatorname{sgn}(\pi_1\pi_2)\prod_{l=1}^{r}\Theta_l((\sigma\pi_1)^{-1}I|_{K_l},(\tau\pi_2)^{-1}J|_{K_l})\right).\end{aligned}$$

Note that for any $\sigma\in\mathfrak{S}_k$ the tuples $\sigma^{-1}I|_{K_l}$ and $\sigma^{-1}J|_{K_l}$ depend only on how σ acts on K_l. Recall also that $\mathfrak{S}(K) = \mathfrak{S}(K_1)\times\ldots\times\mathfrak{S}(K_r)$. Using this for any $\sigma,\tau\in\mathfrak{S}_k$ we obtain

$$\begin{aligned}&\sum_{\pi_1,\pi_2\in\mathfrak{S}(K)}\prod_{l=1}^{r}\Big(\operatorname{sgn}(\pi_1\pi_2|_{K_l})\Theta_l((\sigma\pi_1)^{-1}I|_{K_l},(\tau\pi_2)^{-1}J|_{K_l})\Big) =\\ &\qquad = \prod_{l=1}^{r}\operatorname{symdet}\left([x_{i_{\sigma(s)}l}\partial_{j_{\tau(t)}l}]_{s,t\in K_l}\right) = \prod_{l=1}^{r}\Psi_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l}).\end{aligned}$$

Therefore, we get the following expression for $\Delta(I,J)$:

$$\Delta(I,J)=\sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)}\operatorname{sgn}(\sigma\tau)\prod_{l=1}^{r}\Psi_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l}).$$

Remark 4.5. The summation here runs over all the left cosets of $\mathfrak{S}(K)$ in $\mathfrak{S}_k$. Note that the expression $\operatorname{sgn}(\sigma\tau)\prod_{l=1}^{r}\Psi_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l})$ does not depend on the choice of representatives of $[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)$. Indeed, by (4.5) for any $\pi_1,\pi_2\in\mathfrak{S}(K)$ we have

$$\prod_{l=1}^{r}\Psi_l((\sigma\pi_1)^{-1}I|_{K_l},(\tau\pi_2)^{-1}J|_{K_l})=\prod_{l=1}^{r}\operatorname{sgn}(\pi_1\pi_2|_{K_l})\cdot\prod_{l=1}^{r}\Psi_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l})=$$
$$=\operatorname{sgn}(\pi_1\pi_2)\cdot\prod_{l=1}^{r}\Psi_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l}).$$

Now we apply Lemma 4.3 in order to compute for every $l\in R$ the symmetrized determinant $\Psi_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l})$. Firstly, observe that $\Psi_l(I_l(\sigma),J_l(\tau))=0$ unless there exists a permutation $\pi_l\in\mathfrak{S}(K_l)$ and an element $s_l\in K_l$ such that $j_{\tau(\pi(s))}=i_{\sigma(s)}$ for all $s\in K_l\setminus\{s_l\}$. Moreover, in this case we have

$$\Psi_l(\sigma^{-1}I|_{K_l},\tau^{-1}J|_{K_l})=(-1)^{m_l-1}(m_l-1)!\times$$
$$\times\sum_{\pi_l\in\mathfrak{S}(K_l)}\sum_{s_l\in K_l}\mathbb{1}(\sigma^{-1}I|_{K_l\setminus\{s_l\}}=(\tau\pi_l)^{-1}J|_{K_l\setminus\{s_l\}})\cdot\operatorname{sgn}(\pi_l)x_{I(\sigma(s_l)),l}\partial_{J(\tau\pi_l(s_l)),l}.$$

Combining everything together we get

$$\Delta(I,J)=\sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)}\operatorname{sgn}(\sigma\tau)\prod_{l=1}^{r}\Psi_l(\sigma^{-1}I|_{K_l},(\tau\pi_l)^{-1}J|_{K_l})=\prod_{l\in R}(-1)^{m_l-1}(m_l-1)!\times$$
$$\times\sum_{[\sigma],[\tau]\in\mathfrak{S}_k/\mathfrak{S}(K)}\operatorname{sgn}(\sigma\tau)\sum_{\substack{s_l\in K_l\\ \pi_l\in\mathfrak{S}(K_l)\\ l\in R}}\prod_{l\in R}\Big(\mathbb{1}(\sigma^{-1}I|_{K_l\setminus\{s_l\}}=(\tau\pi_l)^{-1}J|_{K_l\setminus\{s_l\}})\operatorname{sgn}(\pi_l)\,x_{(\sigma^{-1}I)(s_l),l)}\partial_{(\tau^{-1}J)(\pi_l(s_l)),l}\Big).$$

Now observe that $\mathfrak{S}(K)=\prod_{l\in R}\mathfrak{S}(K_l)$. Hence, when τ runs over $\mathfrak{S}_k/\mathfrak{S}(K)$ and π_l runs over $\mathfrak{S}(K_l)$ for each $l\in R$ the product $\tau\cdot\prod_{l\in R}\pi_l$ runs over $\mathfrak{S}_k$. Therefore, we can rewrite the formula above using the summation over $\tau\in\mathfrak{S}_k$ as follows:

$$\Delta(I,J)=\prod_{l\in R}(-1)^{m_l-1}(m_l-1)!\times$$
$$\times\sum_{[\sigma]\in\mathfrak{S}_k/\mathfrak{S}(K)}\operatorname{sgn}(\sigma)\sum_{\substack{s_l\in K_l\\ l\in R}}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(\sigma^{-1}I|_{[k]\setminus\{s_l\colon l\in R\}}=\tau^{-1}J|_{[k]\setminus\{s_l\colon l\in R\}})\cdot\operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{(\sigma^{-1}I)(s_l),l}\partial_{J(\tau(s_l)),l}\Big).$$

Replacing τ with $\tau\sigma$ and introducing the summation over $\sigma\in\mathfrak{S}_k$ instead of the summation over $[\sigma]\in\mathfrak{S}_k/\mathfrak{S}(K)$ yields

$$\Delta(I,J)=(-1)^{k-\ell}\prod_{l\in R}m_l^{-1}\times$$
$$\times\sum_{\sigma\in\mathfrak{S}_k}\sum_{\substack{s_l\in K_l\\ l\in R}}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus\{\sigma(s_l)\colon l\in R\}}=\tau^{-1}J|_{[k]\setminus\{\sigma(s_l)\colon l\in R\}})\cdot\operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{I(\sigma(s_l)),l}\partial_{(\tau^{-1}J)(\sigma(s_l)),l}\Big).$$

For a given ℓ-tuple $\{s_l\}_{l\in R}$ as σ runs over $\mathfrak{S}_k$ the tuple $\{\sigma(s_l)\}_{l\in R}$ runs over all ℓ-tuples in $[k]^{\underline{\ell}}$ and each of them occurs exactly $(k-\ell)!$ times. Since there are precisely $\prod_{l\in R}m_l$ tuples $\{s_l\}_{l\in R}$ in $\prod_{l\in R}R_l$ we have

$$\Delta(I,J)=(-1)^{k-\ell}(k-\ell)!\times$$
$$\times\sum_{U\in[k]^{\underline{R}}}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}}=\tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}})\cdot\operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{I(u_l),l}\partial_{(\tau^{-1}J)(u_l),l}\Big).$$

Here by $[k]^{\underline{R}}$ we denote the set of all q-tuples $U=\{u_l\}_{l\in R}$ of distinct elements in $[k]$. Let $\mathfrak{S}(U)$ be the permutation group of the ℓ-element set $\{u_l : l\in R\}$. Then, we can rewrite the last sum as follows:

$$\Delta(I,J)=\frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\times$$
$$\times\sum_{U\in[k]^{\underline{R}}}\sum_{\pi\in\mathfrak{S}(U)}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}}=\tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}})\cdot\operatorname{sgn}(\tau)\prod_{l\in R}\Big(x_{I(\pi(u_l)),l}\partial_{(\tau^{-1}J)(\pi(u_l)),l}\Big).$$

Clearly, for any $\pi \in \mathfrak{S}(U)$ we have

$$\mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}) = \mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = (\tau\pi)^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}}). \tag{4.6}$$

Therefore, substituting $\tau \mapsto \tau\pi^{-1}$ we obtain

$$\Delta(I,J) = \frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\times$$
$$\times \sum_{U\in[k]^{\underline{R}}}\sum_{\pi\in\mathfrak{S}(U)}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}})\cdot\operatorname{sgn}(\tau\pi)\prod_{l\in R}x_{I(\pi(u_l)),l}\partial_{(\tau^{-1}J)(u_l),l} =$$
$$= \frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\sum_{U\in[k]^{\underline{R}}}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus\{u_l\colon l\in R\}} = \tau^{-1}J|_{[k]\setminus\{u_l\colon l\in R\}})\operatorname{sgn}(\tau)\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\prod_{l\in R}\partial_{(\tau^{-1}J)(u_l),l}.$$

Similarly, note that

$$\sum_{U\in[k]^{\underline{R}}}\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\prod_{l\in R}\partial_{(\tau^{-1}J)(u_l),l} = \frac{1}{\ell!}\sum_{U\in[k]^{\underline{R}}}\sum_{\pi\in\mathfrak{S}(U)}\det([x_{I(\pi(u_\alpha)),\beta}]_{\alpha,\beta\in R})\prod_{l\in R}\partial_{(\tau^{-1}J)(\pi(u_l)),l} =$$
$$= \frac{1}{\ell!}\sum_{U\in[k]^{\underline{R}}}\sum_{\pi\in\mathfrak{S}(U)}\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\operatorname{sgn}(\pi)\prod_{l\in R}\partial_{(\tau^{-1}J)(\pi(u_l)),l} =$$
$$= \frac{1}{\ell!}\sum_{U\in[k]^{\underline{R}}}\det([x_{I(u_\alpha),\beta}]_{\alpha,\beta\in R})\cdot\det([\partial_{(\tau^{-1}I)(u_\alpha),\beta}]_{\alpha,\beta\in R}) =$$
$$= \sum_{V\in\binom{[k]}{\ell}}\det([x_{I(v),l}]_{v\in V,l\in R})\cdot\det([\partial_{(\tau^{-1}J)(v),l}]_{v\in V,l\in R}).$$

Plugging this into the previous formula and taking into account (4.6) gives

$$\Delta(I,J) = \frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\times \tag{4.7}$$
$$\times\sum_{V\in\binom{[k]}{\ell}}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus V} = \tau^{-1}J|_{[k]\setminus V})\operatorname{sgn}(\tau)\det([x_{I(v),l}]_{v\in V,l\in R})\cdot\det([\partial_{(\tau^{-1}J)(v),l}]_{v\in V,l\in R})$$
$$= \frac{(-1)^{k-\ell}(k-\ell)!}{\ell!}\times\sum_{V\in\binom{[k]}{\ell}}\sum_{\tau\in\mathfrak{S}_k}\mathbb{1}(I|_{[k]\setminus V} = J\circ\tau|_{[k]\setminus V})\operatorname{sgn}(\tau)\det([x_{I(v),l}]_{v\in V,l\in R})\cdot\det([\partial_{J(\tau(v)),l}]_{v\in V,l\in R})$$

Now observe that for any given subset $V\subset[k]$ with $|V| = \ell$ there exist either $\ell!$ permutations $\tau\in\mathfrak{S}_k$ such that

$$I|_{[k]\setminus V} = J\circ\tau|_{[k]\setminus V}, \tag{4.8}$$

or none. Indeed, suppose such a permutation τ_0 exists. Then, any other permutation τ that satisfies (4.8) is of the form $\tau = \tau_0\pi$, where $\pi\in\mathfrak{S}(V)$. In particular,

$$\operatorname{sgn}(\tau)\det([\partial_{J(\tau(v)),l}]_{v\in V,l\in R} = \operatorname{sgn}(\tau_0\pi)\operatorname{sgn}(\pi)\det([\partial_{J(\tau_0(v)),l}]_{v\in V,l\in R} = \operatorname{sgn}(\tau_0)\det([\partial_{J(\tau_0(v)),l}]_{v\in V,l\in R}$$

Note that τ satisfying (4.8) exists if and only if all elements of $I|_{\sigma([k]\setminus V)}$ are contained in J. Hence, the summation over τ in (4.7) in fact contains $\ell!$ equal summands. This allows us to write the final formula for $\Delta(I,J)$:

$$\Delta(I,J) = (-1)^{k-\ell}(k-\ell)!\sum_{V,W\in\binom{[k]}{q}}\varepsilon(I,J,I(V),J(W))\det(X_{I(V),R})\cdot\det(D_{J(W),R}). \tag{4.9}$$

Indeed, to get the (4.9) from (4.7) we just note that $\mathbb{1}(I|_{[k]\setminus V} = J\circ\tau|_{[k]\setminus V})\operatorname{sgn}(\tau)$ equals $\varepsilon(I,J,I(V),J(W))$ if $W = \tau(V)$ and τ satisfies (4.8), and is zero otherwise. □

Lemma 4.5. *For any $I,J\in[n]^{\underline{k}}$ we have*

$$\operatorname{symdet} L(E)_{IJ} = \sum_{\ell=0}^{k}(-1)^{k-\ell}(k-\ell)!\,\ell!\left\{{k \atop \ell}\right\}\sum_{R\in\binom{[r]}{\ell}}\sum_{V,W\in\binom{[k]}{\ell}}\varepsilon(I,J,I(V),J(W))\det(X_{I(V),R})\det(D_{J(W),R}).$$

Here by $\left\{{k \atop \ell}\right\}$ we denote the Stirling number of the second kind, i.e. the number of ways to split a k-element set into ℓ non-empty subsets.

Proof. We start with the identity

$$\operatorname{symdet}(L(E))_{IJ} = \sum_{a_1,\dots,a_k=1}^{n}\sum_{\sigma,\tau\in\mathfrak{S}_k}\operatorname{sgn}(\sigma\tau)\cdot x_{i_{\sigma(1)}a_1}\partial_{j_{\tau(1)}a_1}\dots x_{i_{\sigma(k)}a_k}\partial_{j_{\tau(k)}a_k}.$$

For each choice of $a_1, \dots, a_k \in \{1, \dots, r\}$ we rewrite the products using Lemma 4.4. It remains to notice that for any given $R \in \binom{r}{q}$ the number of sequences $(a_1, \dots, a_k) \in [r]^k$ such that $\{a_1, \dots, a_k\} = R$ equals $\left\{{k \atop q}\right\} \cdot q!$ $\square$

4.2. **Proof of the main formulas.** Now we are ready to prove Theorem 3.8 and Corollary 3.9.

Proof of Theorem 3.8. By Proposition 3.6 we have

$$M_{p,q}(Y) = \sum_{\substack{I_1, J_1 \in \binom{[n]}{p} \\ I_2, J_2 \in \binom{[n]}{q} \\ I_1 \sqcup I_1 = J_1 \sqcup J_2}} \operatorname{sgn}\begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix} \det Y_{I_1 J_1} \cdot \operatorname{symdet} L(E)_{J_2 I_2}.$$

Applying Lemma 4.5 to $\operatorname{symdet} L(E)_{J_2 I_2}$ gives the required identity (3.6). $\square$

Proof of Corollary 3.9. Indeed, the identity (3.8) is a direct consequence of (3.6) and the fact that the expression $\operatorname{sgn}\begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix}\varepsilon(J_2, I_2, J_2(V), I_2(W))$, if non-zero, equals $\operatorname{sgn}\begin{pmatrix} I_1 & I_2(W) \\ J_1 & J_2(V) \end{pmatrix}$. The coefficient $\binom{n-p-\ell}{q-\ell}$ appears as the number of ways to choose a $(q-\ell)$-element subset $I_2 \setminus I_1(W) = J_1 \setminus J_2(V)$ in the complement of the $(p+\ell)$-element subset $I_1 \sqcup J_1 = I_2 \sqcup J_2$.

Then, it follows from (3.8) that the difference $M_{p,q} - q! \cdot F_{p,q}$ is a linear combination of $F_{p,\ell}$ with $\ell < q$. Therefore, sets $\{M_{p,q}\}_{p+q \le n}$ and $\{F_{p,q}\}_{p+q \le n}$ generate the same algebra inside $\mathscr{C}(\mathcal{P}(n,r))$. $\square$

5. Capelli's identities and their variations

5.1. **Classical Capelli's identity.** Assume that $r = n$, i.e. that $\mathcal{P}(n,r)$ is a polynomial ring $\mathbb{C}[\mathrm{Mat}(n,n)]$ in n^2 variables x_{ij}, $1 \le i,j \le n$. Recall that $\mathcal{PD}(n,r)$ is the algebra of differential operators on $\mathcal{P}(n,r)$ with polynomial coeficients. Define an element $\Pi \in U(\mathfrak{gl}_n)$ as

$$\Pi = \operatorname{rdet}(E_{ij} + (i-1)\delta_{ij})_{i,j=1}^n = \operatorname{rdet}\begin{bmatrix} E_{11}+0 & E_{12} & \dots & E_{1k} \\ E_{21} & E_{22}+1 & \dots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{k1} & E_{k2} & \dots & E_{kk}+k-1 \end{bmatrix}.$$

The next statement was observed by Alfredo Capelli [1, Capelli] and was used by Hermann Weyl in his treatment of invariant theory for GL_n [13, Weyl].

Proposition 5.1 (Capelli's identity)**.** *The image of* Π *in* $\mathcal{PD}(n,r)$ *equals*

$$L(\Pi) = \det(X)\det(D). \tag{5.1}$$

In particular, the expression on the left is a GL_n*-invariant differential operator.*

Remark 5.1. Since $L(E) = XD^T$ one can rewrite this identity as $\operatorname{rdet}(XD^T + Q) = \det(X) \cdot \det(D^T)$, where Q is the diagonal matrix with entries $(0, 1, \dots, n-1)$. In other words, the Capelli's identity resembles the identity $\det(AB) = \det(A) \cdot \det(B)$ for matrices with commuting entries. However, since $U(\mathfrak{gl}_n)$ is non-commutative and one has to introduce the *quantum correction* Q to get a valid equality.

Example 5.1. For instance, if $n = r = 2$, then this is equivalent to

$$\operatorname{rdet}\begin{bmatrix} x_{11}\partial_{11} + x_{12}\partial_{12} + 0 & x_{11}\partial_{21} + x_{12}\partial_{22} \\ x_{21}\partial_{11} + x_{22}\partial_{12} & x_{21}\partial_{21} + x_{22}\partial_{22} + 1 \end{bmatrix} = \det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \cdot \det\begin{bmatrix} \partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22}. \end{bmatrix}$$

One can verify this by direct computations. Indeed, the left-hand side equals

$$\begin{aligned}
\mathrm{LHS} &= (x_{11}\partial_{11} + x_{12}\partial_{12})(x_{21}\partial_{21} + x_{22}\partial_{22} + 1) - (x_{11}\partial_{21} + x_{12}\partial_{22})(x_{21}\partial_{11} + x_{22}\partial_{12}) = \\
&= x_{11}x_{21}\partial_{11}\partial_{21} + x_{11}x_{22}\partial_{11}\partial_{22} + x_{12}x_{21}\partial_{12}\partial_{21} + x_{12}x_{22}\partial_{12}\partial_{22} + x_{11}\partial_{11} + x_{12}\partial_{12} - \\
&- (x_{11}x_{21}\partial_{21}\partial_{11} + x_{11}\partial_{11} + x_{11}x_{22}\partial_{21}\partial_{12} + x_{12}x_{21}\partial_{22}\partial_{11} + x_{12}x_{22}\partial_{22}\partial_{12} + x_{12}\partial_{12}) = \\
&= x_{11}x_{22}\partial_{11}\partial_{22} + x_{12}x_{21}\partial_{12}\partial_{21} - x_{11}x_{22}\partial_{12}\partial_{21} - x_{12}x_{21}\partial_{11}\partial_{22} = \\
&= (x_{11}x_{22} - x_{12}x_{21})(\partial_{11}\partial_{22} - \partial_{12}\partial_{21}) = \det\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \cdot \det\begin{bmatrix} \partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22}. \end{bmatrix}.
\end{aligned}$$

5.2. **Capelli's identities for rectangular matrices.** Now let r be an arbitrary positive integer. It turns out that one can generalize the Capelli's identity for all minors of the matrix E.

Firstly, for arbitrary k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ in $[n]^k$ introduce the following element $\Pi_{IJ} \in U(\mathfrak{gl}_n)$:

$$\Pi_{IJ} = \operatorname{rdet}[E_{i_\alpha j_\beta} + (\alpha - 1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.$$

For example, if $I = J = (i_1, \dots, i_k) \in [n]^{\underline{k}}$, then

$$\Pi_{II} = \operatorname{rdet} \begin{bmatrix} E_{i_1 i_1} + 0 & E_{i_1 i_2} & \dots & E_{i_1 i_k} \\ E_{i_2 i_1} & E_{i_2 i_2} + 1 & \dots & E_{i_2 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i_k i_1} & E_{i_k i_2} & \dots & E_{i_k i_k} + k - 1 \end{bmatrix},$$

which coincides with Π above when $I = (1, 2, \dots, n)$.

Proposition 5.2. *The following equality holds in $U(\mathfrak{gl}_n)$:*

$$\operatorname{rdet}(E_{ij} + (i - 1 - z)\delta_{ij})_{i,j=1}^n = \sum_{k=0}^{n} (-1)^k z^{\underline{k}} \cdot C_{n-k},$$

where $z^{\underline{k}} = z(z-1)\dots(z-k+1)$ and

$$C_k = \sum_{I \in \binom{[n]}{k}} \Pi_{II}.$$

Moreover, the elements C_k belong to the center $Z(\mathfrak{gl}_n)$ of the universal enveloping algebra $U(\mathfrak{gl}_n)$ and their images in $\mathcal{PD}(n, r)$ are as follows:

$$L(C_k) = \sum_{I \in \binom{[n]}{k}} \sum_{K \in \binom{[r]}{k}} \det(X_{IK}) \det(D_{IK}) \tag{5.2}$$

Remark 5.2. In formula (5.2) we regard $K \in \binom{[r]}{k}$ as an element of $\mathfrak{S}_k \backslash [r]^{\underline{k}}$, i.e. as k-tuple with arbitrarily chosen ordering of elements. Note that the term $\det(X_{IK}) \det(D_{IK})$ does not depend on this choice, so the expression on the right is well defined.

Remark 5.3. The elements C_k are often called *Capelli generators*. Moreover, $Z(\mathfrak{gl}_n)$ is a free algebra generated by $C_1, \dots, C_n$. The classical counterparts of Capelli generators are the coefficients c_k (see (3.1)) of the characteristic polynomial of $Y \in \mathfrak{gl}_n$. The elements $\{c_k\}_{k=1}^n$ are the generators of $S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$.

5.3. **Proofs of Propositions 5.1 and 5.2.** In order to prove Capelli's identity and its variants we prove a more general version for arbitrary minor Π_{IJ}. The following statement can be viewed as a non-commutative analogue of Cauchy-Binet identity from linear algebra.

Proposition 5.3 (Cauchy-Binet type identity)**.** *For any $I, J \in [n]^k$ the image of Π_{IJ} in $\mathcal{PD}(n, r)$ under the map L equals*

$$L(\Pi_{IJ}) = \sum_{K \in \binom{[r]}{k}} \det(X_{IK}) \det(D_{JK}). \tag{5.3}$$

Remark 5.4. Note that the right-hand side of (5.3) is skew-symmetric in entries of I and J and hence, the same holds for $L(\Pi_{IJ})$. From the definition of Π_{IJ} it is clear that Π_{IJ} is skew-symmetric in J (as row determinant), but the skew-symmetricity in I is not immediate from this definition. Formula (5.3) in particular implies $L(\Pi_{IJ})$ is zero whenever I or J contains equal elements.

Proof. Induct on k. For $k = 1$ the identity follows immediately from the definition of $L(E_{ij})$. Now assume that $k > 1$. Consider any k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$. Denote $I' = (i_1, \dots, i_{k-1})$ and $J^{(l)} = (j_1, \dots, \widehat{j_l}, \dots, j_k)$ for every $l \in [k]$. Expanding Π_{IJ} along the k-th row yields

$$\Pi_{IJ} = \sum_{l=1}^{k} (-1)^{k-l} \Pi_{I' J^{(l)}} \cdot (E_{i_k j_l} + (k-1)\delta_{i_k j_l}),$$

and hence by the inductive hypothesis,

$$L(\Pi_{IJ}) = \sum_{l=1}^{k} (-1)^{k-l} L(\Pi_{I' J^{(l)}}) \cdot \left(\sum_{\alpha \in [r]} x_{i_k \alpha} \partial_{j_l \alpha} + (k-1)\delta_{i_k j_l} \right) =$$

$$= \sum_{l=1}^{k} (-1)^{k-l} \sum_{K' \in \binom{[r]}{k-1}} \det(X_{I'K'}) \det(D_{J^{(l)} K'}) \cdot \left(\sum_{\alpha \in [r]} x_{i_k \alpha} \partial_{j_l \alpha} + (k-1)\delta_{i_k j_l} \right).$$

Using the identity $\det(D_{J^{(l)}K'})x_{i_k\alpha} = x_{i_k\alpha}\det(D_{J^{(l)}K'}) + [\det(D_{J^{(l)}K'}), x_{i_k\alpha}]$ we can rewrite the last sum as follows:

$$
\begin{aligned}
L(\Pi_{IJ}) = &\sum_{l=1}^{k}\sum_{\alpha\in[r]}\sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l}\det(X_{I'K'})x_{i_k\alpha}\cdot\det(D_{J^{(l)}K'})\partial_{j_l\alpha}+\\
&+(k-1)\sum_{l=1}^{k}\sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l}\det(X_{I'K'})\cdot\det(D_{J^{(l)}K'})\delta_{i_kj_l}+\\
&+\sum_{l=1}^{k}\sum_{\alpha\in[r]}\sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l}\det(X_{I'K'})\cdot[\det(D_{J^{(l)}K'}), x_{i_k\alpha}]\partial_{j_l\alpha}
\end{aligned}
\tag{5.4}
$$

Denote the three summands on the right-hand side of (5.4) by $S^{(1)}_{IJ}$, $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$, respectively. Observe that $S^{(1)}_{IJ}$ equals

$$
\begin{aligned}
S^{(1)}_{IJ} = &\sum_{l=1}^{k}\sum_{\alpha\in[r]}\sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l}\det(X_{I'K'})x_{i_k\alpha}\cdot\det(D_{J^{(l)}K'})\partial_{j_l\alpha} =\\
&= \sum_{\alpha\in[r]}\sum_{K'\in\binom{[r]}{k-1}} \det(X_{I'K'})x_{i_k\alpha}\cdot\det(D_{JK'_{(\alpha)}}) = \sum_{K\in\binom{[r]}{k}}\det(X_{IK})\cdot\det(D_{JK}).
\end{aligned}
$$

Here by $K'_{(\alpha)}$ we denote a k-tuple in $[n]^{\underline{k}}$ whose first $k-1$ entries coincide with those of K' and whose k-th entry equals α. (Recall that K' can be viewed as a $(k-1)$-tuple.) The last equality follows the cofactor expansion of $\det(X_{IK})$ along the last row.

Therefore, it remains to show that the sum $S^{(2)}_{IJ} + S^{(3)}_{IJ}$ is zero. We consider three cases depending on the number q of occurences of i_k in the k-tuple J.

Case 1. $q = 0$. Then, for any $l \in [k]$ and $\alpha \in [r]$ we have $\delta_{i_kj_l} = 0$ and $[\det(D_{J^{(l)}K'}), x_{i_k\alpha}] = 0$. Hence, both $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$ are zero which concludes the proof.

In the remaining two cases we use the following lemma.

Lemma 5.4. *For any k-tuples $I = (i_1, \dots, i_k) \in$ and $J = (j_1, \dots, j_k)$ in $[n]^k$ and any i and j, we have*

$$
\sum_{\alpha\in[r]}[\det(D_{IJ}), x_{i\alpha}]\partial_{j\alpha} = \begin{cases} 0, & \text{if } I \notin [n]^{\underline{k}} \text{ or } i \notin I.\\ \det(D_{KJ}), & \text{if } i_p = i \text{ for } p \in [k], \end{cases}
$$

where in the second case we put $K = (i_1, \dots, i_{p-1}, j, i_{p+1}, \dots, i_k)$.

Proof. If I contains equal entries or $i \notin I$, then it clear that $[\det(D_{IJ}), x_{i\alpha}] = 0$ for all $\alpha \in [r]$. Otherwise, if $p \in [k]$ is such that $i_p = i$, then $[\det(D_{IJ}), x_{i\alpha}]\partial_{j\alpha}$ is zero if $\alpha \notin J$ and equals to the cofactor of the element $\partial_{j\alpha}$ of the matrix D_{KJ} otherwise. Thus, $\sum_{\alpha\in[r]}[\det(D_{IJ}), x_{i\alpha}]\partial_{j\alpha} = \det(D_{KJ})$, as claimed. □

Case 2. $q = 1$. Let $p \in [k]$ be such that $i_k = j_p$. Observe that both sides of (5.3) are skew-symmetric with respect to J. Thus, we may assume without loss of generality that $p = k$. In this case we can rewrite $S^{(2)}_{IJ}$ as follows:

$$
S^{(2)}_{IJ} = (k-1)\sum_{l=1}^{k}\sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l}\det(X_{I'K'})\cdot\det(D_{J^{(l)}K'})\delta_{i_kj_l} = (k-1)\sum_{K'\in\binom{[r]}{k-1}}\det(X_{I'K'})\cdot\det(D_{J^{(k)}K'}).
$$

Applying Lemma 5.4 gives the following expression for the $S^{(3)}_{IJ}$:

$$
\begin{aligned}
S^{(3)}_{IJ} = &\sum_{l=1}^{k}\sum_{\alpha\in[r]}\sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l}\det(X_{I'K'})\cdot[\det(D_{J^{(l)}K'}), x_{i_k\alpha}]\partial_{j_l\alpha} =\\
&= \sum_{l\in[k-1]}\sum_{K'\in\binom{[r]}{k-1}} (-1)^{k-l}\det(X_{I'K'})\cdot(-1)^{k-l-1}\det(D_{J^{(k)}K'}) =\\
&= -(k-1)\sum_{K'\in\binom{[r]}{k-1}}\det(X_{I'K'})\cdot\det(D_{J^{(k)}K'}).
\end{aligned}
$$

Therefore, the $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$ add up to zero, as claimed.

Case 3. $q > 1$. Let us show that in this case both the $S^{(2)}_{IJ}$ an $S^{(3)}_{IJ}$ vanish. If $q \geq 3$, then each $(k-1)$-tuple $J^{(l)}$, $l = 1, \dots, k$, contains at least two entries that are equal to i_k. Hence, all terms in sums $S^{(2)}_{IJ}$ and $S^{(3)}_{IJ}$ are zero since $\det(D_{J^l K'}) = 0$.

Now assume that $q = 2$. Similar to the second case we may assume that $j_{k-1} = j_k = i_k$ and $j_l \neq i_k$ for $l < k-1$. Then, for $S^{(2)}_{IJ}$ we have

$$S^{(2)}_{IJ} = (k-1)\sum_{l=1}^{k} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot \det(D_{J^{(l)}K'})\delta_{i_k j_l} =$$

$$= (k-1) \sum_{K' \in \binom{[r]}{k-1}} \Big(-\det(X_{I'K'}) \cdot \det(D_{J^{(k-1)}K'}) + \det(X_{I'K'}) \cdot \det(D_{J^{(k)}K'}) \Big) = 0$$

because $\det(D_{J^{(l)}K'}) = 0$ for $l < k-1$ while $J^{(k-1)} = J^{(k)}$. Lemma 5.4 implies that $S^{(3)}_{IJ}$ equals

$$S^{(3)}_{IJ} = \sum_{l=1}^{k} \sum_{\alpha \in [r]} \sum_{K' \in \binom{[r]}{k-1}} (-1)^{k-l} \det(X_{I'K'}) \cdot [\det(D_{J^{(l)}K'}), x_{i_k \alpha}]\partial_{j_l \alpha} =$$

$$= \sum_{K' \in \binom{[r]}{k-1}} \det(X_{I'K'}) \sum_{\alpha \in [r]} \Big(-[\det(D_{J^{(k-1)}K'}), x_{i_k\alpha}]\partial_{j_{k-1}\alpha} + [\det(D_{J^{(k)}K'}), x_{i_k\alpha}]\partial_{j_k\alpha} \Big) = 0$$

since $J^{(k-1)} = J^{(k)}$ and $j_{k-1} = j_k = i_k$.

Therefore, if $q > 1$, we have $S^{(2)}_{IJ} = S^{(3)}_{IJ} = 0$ which concludes the proof of the proposition. ☐

Corollary 5.5. *For any $I, J \in [n]^k$ and any $\sigma, \tau \in \mathfrak{S}_k$ we have $\Pi_{\sigma I, \tau J} = \operatorname{sgn}(\sigma\tau) \cdot \Pi_{IJ}$.*

Proof. By the previous proposition, we have the identity $L(\Pi_{\sigma I,\tau J}) = \operatorname{sgn}(\sigma\tau) \cdot L(\Pi_{IJ})$. By Proposition 2.2, for $r = n$ the map L faithfully maps $U(\mathfrak{gl}_n)$ to $\mathcal{PD}(n, r)$, hence the result follows. ☐

Corollary 5.6. *For any k-tuples $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ in $[n]^k$ we have*

$$\Pi_{IJ} = \operatorname{rdet}[E_{i_\alpha j_\beta} + (\alpha - 1)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1} = \operatorname{cdet}[E_{i_\alpha j_\beta} + (k - \alpha)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1}.$$

Proof. We prove that the images of Π_{IJ} and $\operatorname{cdet}[E_{i_\alpha j_\beta} + (k-\alpha)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1}$ under the map L coincide. This is sufficient since by Proposition 2.2, for $r = n$ the map $L \colon U(\mathfrak{gl}_n) \to \mathcal{PD}(n, r)$ is injective.

In view of formula (5.3), the equality $L(\Pi_{IJ}) = \operatorname{cdet}[L(E_{i_\alpha j_\beta}) + (k-\alpha)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1}$ is equivalent to

$$\operatorname{cdet}[L(E_{i_\alpha j_\beta}) + (k-\alpha)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1} = \sum_{K \in \binom{[r]}{k}} \det(X_{IK}) \det(D_{JK}).$$

This identity can be proven by induction on k. In fact, the proof is essentially the same as in Proposition 5.3, so we only give an outline.

The base case $k = 1$ follows from the definition of $L(E_{ij})$. Assume that $k > 1$. To perform the inductive step, we expand the column determinant along the first column which together with the inductive assumption gives that

$$\operatorname{cdet}[L(E_{i_\alpha j_\beta}) + (k-\alpha)\delta_{i_\alpha j_\beta}]^k_{\alpha,\beta=1} = \sum_{l=1}^{k} (-1)^{k-l} \left(\sum_{\alpha \in [r]} x_{i_l \alpha}\partial_{j_1 \alpha} + (k-1)\delta_{i_k j_l} \right) \sum_{K' \in \binom{[r]}{k-1}} \det(X_{I^{(l)}K'}) \det(D_{J'K'}),$$

where $I^{(l)} = (i_1, \dots, \widehat{i_l}, \dots, i_k)$ and $J' = (j_2, \dots, j_k)$. Rewriting this expression in its normal form (use commutation relations in Weyl algebras as in Proposition 5.3) finishes the inductive step. ☐

Remark 5.5. One can prove this identity directly in $U(\mathfrak{gl}_n)$ using ternary relations and Yang-Baxter equation (see Lemma 6.15 in the next section).

Can one use an automorphism of Weyl algebra that switches each x_{ij} with ∂_{ij} but reverses the order of multiplication?

5.3.1. *Characteristic polynomial of the Capelli matrix.* Define

$$C_k(z) = \sum_{I \in \binom{[n]}{k}} \operatorname{rdet}(E_{i_\alpha i_\beta} + (\alpha - 1 - z)\delta_{ij})_{\alpha,\beta \in [k]}.$$

Observe that $C_k(0) = C_k$.

Proposition 5.7. *For any* $0 \le k \le n$

$$C_k(z) = \sum_{m=0}^{k} (-1)^m \binom{n-k+m}{m} z^{\underline{m}} \cdot C_{k-m}. \tag{5.5}$$

Proof. [5, Howe-Umeda] We induct on k. The cases $k=0$ and $k=1$ are clear. Denote the right hand-side of (5.5) by $B_k(z)$. Let Δ be the difference operator defined as $(\Delta f)(z) = f(z+1) - f(z)$. Since both $B_k(z)$ and $C_k(z)$ are $U(\mathfrak{gl}_n)$-valued polynomials in z and $B_k(0) = C_k(0) = C_k$, it suffices to check that $(\Delta B_k)(z) = (\Delta C_k)(z)$. We have

$$(\Delta B_k)(z) = \sum_{m=0}^{k} (-1)^m \binom{n-k+m}{m} \Delta(z^{\underline{m}}) \cdot C_{k-l} = \sum_{m=1}^{k} (-1)^m m \binom{n-k+m}{m} z^{\underline{m-1}} \cdot C_{k-m}.$$

Note that $m\binom{n-k+m}{m} = (n-k+1)\binom{n-k+m}{m-1}$, and hence we obtain

$$(\Delta B_k)(z) = (n-k+1) \sum_{m=0}^{k-1} (-1)^{m+1} \binom{n-k+m+1}{m} z^{\underline{m}} \cdot C_{k-1-m} = -(n-k+1) B_{k-1}(z).$$

Now let us compute $(\Delta C_k)(z)$. For any k-element subset $I = \{i_1, \ldots, i_k\}$ of $[n]$, where $i_1 < \ldots < i_k$, we set

$$E^{\natural}_{II} = \begin{bmatrix} E_{i_1 i_1} + 0 & E_{i_1 i_2} & \ldots & E_{i_1 i_k} \\ E_{i_2 i_1} & E_{i_2 i_2} + 1 & \ldots & E_{i_2 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i_k i_1} & E_{i_k i_2} & \ldots & E_{i_k i_k} + k - 1 \end{bmatrix} \text{ and } \Pi_{II}(z) = \operatorname{rdet}(E^{\natural}_{II} - z \cdot \mathrm{Id}_k).$$

Then,

$$(\Delta \Pi_{II})(z) = \Pi_{II}(z+1) - \Pi_{II}(z) = \sum_{p=1}^{k} \big(\operatorname{rdet}(E^{\natural}_{II} - (z+1) \cdot \mathrm{Id}_k + \Lambda_{p-1}) - \operatorname{rdet}(E^{\natural}_{II} - (z+1) \cdot \mathrm{Id}_k + \Lambda_p) \big),$$

where Λ_p is a diagonal matrix with first p diagonal entries equal to 1 and the last $k-p$ entries equal to 0. Expanding the row determinant of $E^{\natural} + z \cdot \mathrm{Id}_k + \Lambda_{p-1}$ along the p-th row gives

$$\operatorname{rdet}(E^{\natural}_{II} - (z+1) \cdot \mathrm{Id}_k + \Lambda_{p-1}) = \operatorname{rdet}(E^{\natural}_{II} - (z+1) \cdot \mathrm{Id}_k + \Lambda_p) - \operatorname{rdet}(E^{\natural}_{I^{(p)} I^{(p)}} - z \cdot \mathrm{Id}_{k-1}),$$

where $I^{(p)} = I \setminus \{i_p\}$. Therefore,

$$(\Delta C_k)(z) = \sum_{I \in \binom{[n]}{k}} (\Delta \Pi_{II})(z) = - \sum_{I \in \binom{[n]}{k}} \sum_{p=1}^{k} \Pi_{I^{(p)} I^{(p)}}(z) = -(n-k+1) C_{k-1}(z).$$

By the inductive hypothesis, $B_{k-1}(z) = C_{k-1}(z)$. Therefore, $(\Delta B_k)(z) = (\Delta C_k)(z)$ which concludes the inductive step due to $B_k(0) = C_k(0) = C_k$. □

6. Yang-Baxter equation, Yangian and ternary $RTT = TTR$ relation

Here we review several computational tools related to Yangians and R-matrix formalism. We are essentially following book by Molev (see [8, Section 1] for more details).

6.1. **Notation.** Let $\{e_{ij}\}_{i,j=1}^n$ be the standard matrix units of $\mathrm{Mat}(n,n)$ and let $\{E_{ij}\}_{i,j=1}^n$ be the corresponding generators of the universal enveloping algebra $U(\mathfrak{gl}_n)$. Matrix units $\{e_{ij}\}_{i,j=1}^n$ act on $\mathbb{C}^n$ spanned by $e_1, \ldots, e_n$ in a usual way:

$$e_{ij} e_k = \delta_{jk} e_i.$$

Most of computations will be performed inside algebras of the form

$$U(\mathfrak{gl}_n) \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m}.$$

For any $C = \sum_{i,j=1}^n c_{ij} \otimes e_{ij} \in U(\mathfrak{gl}_n) \otimes \operatorname{End} \mathbb{C}^n$ and any $D = \sum_{i,j,k,l=1}^n d_{ijkl} e_{ij} \otimes e_{kl} \in (\operatorname{End} \mathbb{C}^n)^{\otimes 2}$ (these operators might depend on some parameters) we define

$$C_a = \sum_{i,j=1}^{n} c_{ij} \otimes 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \in U(\mathfrak{gl}_n) \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m},$$

and

$$D_{ab} = \sum_{i,j=1}^{n} d_{ijkl} \cdot 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{kl} \otimes 1^{\otimes(m-b)} \in (\operatorname{End} \mathbb{C}^n)^{\otimes m}.$$

Usually we identify $(\operatorname{End} \mathbb{C}^n)^{\otimes m}$ with the subalgebra $1 \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m}$ inside $U(\mathfrak{gl}_n) \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m}$.

Each element σ of symmetric group $\mathfrak{S}_m$ defines an element of $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$ whose action on $(\mathbb{C}^n)^{\otimes m}$ corresponds to permuting the tensors via σ. Namely, $\sigma \in \mathfrak{S}_m$ corresponds to

$$\sum_{i_1,\ldots,i_m=1}^{n} e_{i_1 i_{\sigma(1)}} \otimes \ldots \otimes e_{i_m i_{\sigma(m)}} \in (\operatorname{End}\mathbb{C}^n)^{\otimes m}.$$

Clearly, this gives rise to an embedding of the group algebra $\mathbb{C}[\mathfrak{S}_m]$ into $(\operatorname{End}\mathbb{C}^n)^{\otimes m}$. We denote by A_m the antisymmetrization operator, i.e.

$$A_m = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma) \sum_{i_1,\ldots,i_m=1}^{n} e_{i_1 i_{\sigma(1)}} \otimes \ldots \otimes e_{i_m i_{\sigma(m)}} \in (\operatorname{End}\mathbb{C}^n)^{\otimes m}.$$

It is not difficult to see that $A_m^2 = m! \cdot A_m$ and $A_m P_{ij} = P_{ij} A_m = -A_m$ for any distinct $i, j \in \{1, \ldots, m\}$.

6.2. **On R-matrices.** Define the *Yang R-matrix*

$$R(u) = 1 - u^{-1} P, \text{ where } P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}.$$

Next, we recall the *Yang-Baxter equation*:

Proposition 6.1 (Yang-Baxter equation)**.** *For any commuting indeterminates u, v and w*

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v). \tag{6.1}$$

We also define for any $m \geq 2$ commuting indeterminates $u_1, \ldots, u_m$ the following rational function:

$$R(u_1, u_2, \ldots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \ldots (R_{1m} \ldots R_{12}), \tag{6.2}$$

where we use shorthand notation $R_{ij} = R_{ij}(u_i - u_j)$.

Remark 6.1. Note that for $m = 1$ this is just Yang R-matrix: $R(u_1, u_2) = R_{12}(u_1 - u_2)$.

There is an alternative definition of $R(u_1, u_2, \ldots, u_m)$.

Lemma 6.2.

$$R(u_1, u_2, \ldots, u_m) = (R_{12} \ldots R_{1m}) \ldots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m})$$

Remark 6.2. Note that for $m = 2$ the statement of the lemma is trivial and for $m = 3$ it is equivalent to the Yang-Baxter equation.

Proof. We need to check the following equality:

$$(R_{12} \ldots R_{1m}) \ldots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m}) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \ldots (R_{1m} \ldots R_{12}).$$

We induct on m. The base case $m = 2$ is trivial, so assume that $m \geq 3$. Then, our aim is to check that

$$(R_{12} \ldots R_{1m})R'(u_2, \ldots, u_m) = R'(u_2, \ldots, u_m)(R_{1m} \ldots R_{12}), \tag{6.3}$$

where

$$R'(u_2, \ldots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \ldots (R_{2m} \ldots R_{23})$$

since by the inductive hypothesis we also have

$$R'(u_2, \ldots, u_m) = (R_{23} \ldots R_{2m}) \ldots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m}).$$

Now we need the following auxiliary identity.

Claim. For any $2 \leq k \leq m$

$$(R_{k,m} \ldots R_{k,k+1})(R_{12} \ldots R_{1,k-1})(R_{1,m} \ldots R_{1,k}) = (R_{12} \ldots R_{1,k+1})(R_{1m} \ldots R_{1,k})(R_{k,m} \ldots R_{k,k+1}).$$

To prove this claim, note first that any two R-matrices with disjoint sets of indices are permutable. Hence, we have

$$\begin{aligned}(R_{k,m} \ldots R_{k,k+1})(R_{12} \ldots R_{1,k-1})(R_{1,m} \ldots R_{1,k}) &= (R_{12} \ldots R_{1,k-1})(R_{k,m} \ldots R_{k,k+1})(R_{1m} \ldots R_{1,k}) = \\ &= (R_{12} \ldots R_{1,k-1})(R_{k,m}R_{1,m}) \ldots (R_{k,k+1}R_{1,k+1})R_{1,k}.\end{aligned}$$

Applying Yang-Baxter equation for products $R_{k,k+j}R_{1,k+j}R_{1,k}$, $j = 1, \ldots, m-k$, we obtain

$$\begin{aligned}(R_{k,m} \ldots R_{k,k+1})(R_{12} \ldots R_{1,k-1})(R_{1,m} \ldots R_{1,k}) &= (R_{12} \ldots R_{1,k-1})R_{1,k}(R_{1,m}R_{k,m}) \ldots (R_{1,k+1}R_{k,k+1}) = \\ &= (R_{12} \ldots R_{1,k})(R_{1m} \ldots R_{1,k+1})(R_{k,m} \ldots R_{k,k+1}),\end{aligned}$$

which concludes the proof of the claim.

Now we apply the claim above for $k=2,\dots,m$ and obtain

$$\begin{aligned}R'(u_2,\dots,u_m)(R_{1m}\dots R_{12}) &= (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1})\dots(R_{2,m}\dots R_{23})(R_{1,m}\dots R_{12}) =\\ &= (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1})\dots(R_{12}R_{1,m}\dots R_{13})(R_{2,m}\dots R_{23}) =\\ &= (R_{m-1,m})\dots(R_{k,m}\dots R_{k,k+1})(R_{12}\dots R_{1,k-1}R_{1,m}\dots R_{1,k})\dots(R_{2,m}\dots R_{23}) =\\ &= (R_{m-1,m})\dots(R_{12}\dots R_{1,k+1}R_{1m}\dots R_{1,k})(R_{k,m}\dots R_{k,k+1})\dots(R_{2,m}\dots R_{23}) =\\ &= (R_{1,2}\dots R_{1m})(R_{m-1,m})(R_{m-2,m}R_{m-2,m-1})\dots(R_{2,m}\dots R_{23}) =\\ &= (R_{1,2}\dots R_{1m})R'(u_2,\dots,u_m),\end{aligned}$$

and hence, (6.3) holds. □

Next we discuss certain specializations of $R(u_1,\dots,u_m)$.

Proposition 6.3. *If $u_i-u_{i+1}=1$ for all $i=1,\dots,m-1$, then*

$$R(u_1,u_2,\dots,u_m)=A_m.$$

Remark 6.3. In fact, if $u_i-u_{i+1}=-1$ for all i, then $R(u_1,\dots,u_m)=H_m$, where H_m is the symmetrizer (but we will not need this fact). Moreover, both identities are particular cases of the so-called *fusion procedure.*

Lemma 6.4. *For any $m\geq 1$ and any commuting indeterminates u and v*

$$A_mR_{0m}(u-v-m+1)\dots R_{01}(u-v)=A_m\left(1-\frac{1}{u-v}(P_{01}+\dots+P_{0m})\right).$$

Similarly,

$$R_{01}(u-v)\dots R_{0m}(u-v-m+1)A_m=\left(1-\frac{1}{u-v}(P_{01}+\dots+P_{0m})\right)A_m.$$

Proof. We have

$$A_mR_{0m}(u-v-m+1)\dots R_{01}(u-v)=A_m\left(1-\frac{1}{u-v-m+1}P_{0m}\right)\dots\left(1-\frac{1}{u-v}P_{01}\right).$$

Now observe that for any distinct indices $i_1,\dots,i_s\in\{1,\dots,m\}$ one has

$$A_mP_{0,i_1}\dots P_{0,i_s}=(-1)^{s-1}A_mP_{0,i_1}.$$

Therefore,

$$\begin{aligned}A_mR_{0m}(u-v-m+1)\dots R_{01}(u-v) &= \sum_{\substack{0\leq s\leq m\\ 1\leq i_s<\dots<i_1\leq m}}\frac{(-1)^s}{(u-v-m+i_1)\dots(u-v-m+i_s)}A_mP_{0,i_1}\dots P_{0,i_s}=\\ &= A_m-\sum_{\substack{1\leq s\leq m\\ 1\leq i_s<\dots<i_1\leq m}}\frac{1}{(u-v-m+i_1)\dots(u-v-m+i_s)}A_mP_{0,i_1}=\\ &= A_m(1-\alpha_1P_{01}\dots-\alpha_mP_{0m}),\end{aligned}$$

where the coefficients α_i are as follows:

$$\begin{aligned}\alpha_i &= \frac{1}{u-v-m+i}\sum_{\substack{1\leq s\leq m\\ 1\leq i_s<\dots<i_2<i}}\frac{1}{(u-v-m+i_2)\dots(u-v-m+i_s)}=\\ &= \frac{1}{u-v-m+i}\left(1+\frac{1}{u-v-m+i-1}\right)\dots\left(1+\frac{1}{u-v}\right)=\frac{1}{u-v}.\end{aligned}$$

Thus,

$$A_mR_{0m}(u-v-m+1)\dots R_{01}(u-v)=A_m\left(1-\frac{1}{u-v}(P_{01}+\dots+P_{0m})\right),$$

as claimed. The second identity can be proved in a simlar way. Alternatively, one can notice that the left-hand side expressions of both identites are both just $R(u,v,v-1,\dots,v-m+1)$ while $P_{01}+\dots+P_{0m}$ commutes with A_m. □

Later we will need the explicit formula for another specialization of $R(u_1,\dots,u_m)$. It will be used for proving the commutativity of Bethe subalgebras (see Subsection 6.4 below).

Let k and l be positive integers. Consider the following specialization of $u_1,\dots,u_k,u_{k+1},\dots,u_{k+l}$:

$$u_i=u-i+1,\ i\in\{1,\dots,k\},\ u_{k+j}=v-j+1,\ j\in\{1,\dots,l\}. \tag{6.4}$$

Now define

$$A_k=R(u_1,\dots,u_k),\ A'_l=R(u_{k+1},\dots,u_{k+l}).$$

In view of Proposition 6.3 these operators are the antisymmetrizers that correspond to sets of indices $\{1,\ldots,k\}$ and $\{k+1,\ldots,k+l\}$, respectively. Observe that $A_kA'_l=A'_lA_k$.

Proposition 6.5. *Define the operator*

$$\widetilde{R}(u,v)=\sum_{p=0}^{\min\{k,l\}}\frac{(-1)^p p!}{(u-v-k+1)\ldots(u-v-k+p)}\sum_{\substack{1\le i_1<\ldots<i_p\le k\\1\le j_1<\ldots<j_p\le l}}P_{i_1,k+j_1}\ldots P_{i_p,k+j_p}.$$

Then, under the specialization (6.4) *we have*

$$R(u_1,\ldots,u_{k+l})=\widetilde{R}(u,v)A_kA'_l=A_kA'_l\widetilde{R}(u,v).$$

Remark 6.4. The exact form of the operator $\widetilde{R}(u,v)$ is not important here (note however that for $k=l=1$ this is just the Yang R-matrix). The important obervation here is that $\widetilde{R}(u,v)$ and $A_kA'_l$ commute and their product equals $R(u_1,\ldots,u_{k+l})$.

Proof. The proof is a bit more complicated version of the argument that was used in Proposition 6.3. Firstly, recall that

$$\begin{aligned}R(u_1,\ldots,u_{k+l})&=(R_{k+l-1,k+l})(R_{k+l-2,k+l}R_{k+l-2,k+l-1})\ldots(R_{1,k+l}\ldots R_{12})=\\&=R(u_{k+1},\ldots,u_{k+l})(R_{k,k+l}\ldots R_{k,k+1})(R_{k-1,k+l}\ldots R_{k-1,k})\ldots(R_{1,k+l}\ldots R_{1,2})\\&=R(u_{k+1},\ldots,u_{k+l})(R_{k,k+l}\ldots R_{k,k+1})\ldots(R_{1,k+l}\ldots R_{1,k+1})(R_{k-1,k})\ldots(R_{1,k}\ldots R_{12})=\\&=R(u_{k+1},\ldots,u_{k+l})(R_{k,k+l}\ldots R_{k,k+1})\ldots(R_{1,k+l}\ldots R_{1,k+1})R(u_1,u_2,\ldots,u_k).\end{aligned}$$

Now we apply l times the identity (6.3) and obtain

$$\begin{aligned}R(u_1,\ldots,u_{k+l})&=R(u_{k+1},\ldots,u_{k+l})(R_{k,k+l}\ldots R_{k,k+1})\ldots(R_{1,k+l}\ldots R_{1,k+1})R(u_1,u_2,\ldots,u_k)=\\&=(R_{k,k+1}\ldots R_{k,k+l})\ldots(R_{1,k+1}\ldots R_{1,k+l})R(u_{k+1},\ldots,u_{k+l})R(u_1,u_2,\ldots,u_k)=\\&=(R_{k,k+1}\ldots R_{k,k+l})\ldots(R_{1,k+1}\ldots R_{1,k+l})A'_lA_k=\\&=(R_{k,k+1}\ldots R_{1,k+1})\ldots(R_{k,k+l}\ldots R_{1,k+l})A_kA'_l.\end{aligned}$$

Using the other formula for $R(u_1,\ldots,u_{k+l})$ one can check similarly that

$$\begin{aligned}R(u_1,\ldots,u_{k+l})&=(R_{12}\ldots R_{1,k+l})\ldots(R_{k+l-2,k+l-1})(R_{k+l-2,k+l-1}R_{k+l-1,k+l})=\\&=A_kA'_l(R_{1,k+l}\ldots R_{1,k+1})\ldots(R_{k,k+l}\ldots R_{k,k+1})=\\&=A_kA'_l(R_{1,k+l}\ldots R_{k,k+l})\ldots(R_{1,k+1}\ldots R_{k,k+1}).\end{aligned}$$

Now we prove that $R(u_1,\ldots,u_{k+l})=\widetilde{R}(u,v)A_kA'_l$. The second identity is completely analogous.

Indeed, applying Lemma 6.4 several times we obtain:

$$\begin{aligned}R(u_1,\ldots,u_{k+l})&=(R_{k,k+1}\ldots R_{1,k+1})\ldots(R_{k,k+l}\ldots R_{1,k+l})A_kA'_l=\\&=(R_{k,k+1}\ldots R_{1,k+1})\ldots(R_{k,k+l-1}\ldots R_{1,k+l-1})\left(1-\frac{1}{u-v-k+l}(P_{1,k+1}+\ldots+P_{k,k+1})\right)A_kA'_l=\\&=(R_{k,k+1}\ldots R_{1,k+1})\ldots(R_{k,k+l-1}\ldots R_{1,k+l-1})A_k\left(1-\frac{1}{u-v-k+l}(P_{1,k+1}+\ldots+P_{k,k+1})\right)A'_l=\\&=A_k\left(1-\frac{1}{u-v-k+1}(P_{1,k+l}+\ldots+P_{k,k+l})\right)\ldots\left(1-\frac{1}{u-v-k+l}(P_{1,k+1}+\ldots+P_{k,k+1})\right)A'_l=\\&=\left(1-\frac{1}{u-v-k+1}(P_{1,k+l}+\ldots+P_{k,k+l})\right)\ldots\left(1-\frac{1}{u-v-k+l}(P_{1,k+1}+\ldots+P_{k,k+1})\right)A'_lA_k.\end{aligned}$$

Observe that for any distinct indices $j_1,\ldots,j_s\in\{1,\ldots,l\}$ and any $i\in\{1,\ldots,k\}$ we have

$$P_{i,k+j_1}\ldots P_{i,k+j_s}A'_l=(-1)^{s-1}P_{i,k+j_s}A'_l.$$

Besides that, if $i_1\neq i_2$ and $j_1\neq j_2$, then

$$P_{i_1,k+j_1}P_{i_2,k+j_2}A'_lA_k=P_{i_2,k+j_1}P_{i_1,k+j_2}A'_lA_k.$$

Using these formulas one obtains that

$$R(u_1,\ldots,u_{k+l})=\sum_{p\ge0}p!\cdot\sum_{\substack{1\le i_1<\ldots<i_p\le k\\1\le j_1<\ldots<j_p\le l}}\alpha_{I,J}\cdot P_{i_1,k+j_1}\ldots P_{i_p,k+j_p},$$

for certain coefficients $\alpha_{I,J}$, where $I=\{i_1,\ldots,i_p\}$ and $J=\{j_1,\ldots,j_p\}$. Here, of course, $p\le\min\{k,l\}$ and the factor $p!$ is due to permutations of $i_1,\ldots,i_p$. Note that $\alpha_{I,J}$ is the sum of products $(-1)^s(u-v-k+b_1)^{-1}\ldots(u-v-k+b_s)^{-1}$ over all sequences $\{(a_i,b_i)\}_{i=1}^s$ which satisfy the following conditions:

- $a_1,\ldots,a_s\in\{1,\ldots,k\}$;

- $l \geq b_1 > \ldots > b_s \geq 1$;
- as a set, $\{a_1, \ldots, a_s\}$ coincides with $I = \{i_1, \ldots, i_p\}$;
- for each $1 \leq q \leq p$ the last r which satisfies $a_r = i_q$ also satisfies $b_r = j_q$.

It remains to compute $\alpha_{I,J}$. The observations above imply that for given $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_p\}$ the corresponding coefficient $\alpha_{I,J}$ equals

$$\alpha_{I,J} = \prod_{j=1}^{j_1-1} \Bigg(1 + \underbrace{\frac{1}{u-v-k+j} + \ldots + \frac{1}{u-v-k+j}}_{p \text{ times}}\Bigg) \cdot \frac{-1}{u-v-k+j_1} \times$$
$$\times \prod_{j=j_1+1}^{j_2-1} \Bigg(1 + \underbrace{\frac{1}{u-v-k+j} + \ldots + \frac{1}{u-v-k+j}}_{p-1 \text{ times}}\Bigg) \cdot \frac{-1}{u-v-k+j_2} \times \ldots \times$$
$$\times \prod_{j=j_{p-1}+1}^{j_p-1} \left(1 + \frac{1}{u-v-k+j}\right) \cdot \frac{-1}{u-v-k+j_p} =$$
$$= (-1)^p \prod_{j=p+1}^{j_p} (u-v-k+j) \prod_{j=1}^{j_p} (u-v-k+j)^{-1} =$$
$$= \frac{(-1)^p}{(u-v-k+1)\ldots(u-v-k+p)}.$$

Comparing this with the definition of $\widetilde{R}(u,v)$ the claim follows. ☐

6.3. **Ternary relation and the Yangian.** Consider an $n \times n$ matrix $T(u)$ defined as follows:

$$T(u) = \sum_{i,j=1}^{n} t_{ij}(u) \otimes e_{ij}, \text{ where } t_{ij}(u) = \delta_{ij} + u^{-1}E_{ij} \in U(\mathfrak{gl}_n).$$

Proposition 6.6 (Ternary relation)**.**

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v). \tag{6.5}$$

Remark 6.5. In fact, this ternary relation (or rather (6.6) below) gives precisely the relations for $\{t_{ij}(u)\}_{i,j=1}^n$ that define the Yangian algebra $Y(\mathfrak{gl}_n)$. Our matrix $T(u)$ is the image of the corresponding matrix for $Y(\mathfrak{gl}_n)$ under the so-called *evaluation homomorphism* $\mathrm{ev}_n \colon Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ which is defined as $\mathrm{ev}_n(t_{ij}(u)) = \delta_{ij} + u^{-1}E_{ij}$.

Proof. Note that both sides are elements of the algebra $U(\mathfrak{gl}_n) \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n$. Therefore, in order to check this relation it suffices to check that their actions on all basis vectors $e_j \otimes e_l$. The left-hand side then yields

$$\sum_{i,k=1}^{n} t_{ij}(u)t_{kl}(v) \otimes e_i \otimes e_k - \frac{1}{u-v} \sum_{i,k=1}^{n} t_{ij}(u)t_{kl}(v) \otimes e_k \otimes e_i,$$

and the right-hand side gives

$$\sum_{i,k=1}^{n} t_{kl}(v)t_{ij}(u) \otimes e_i \otimes e_k - \frac{1}{u-v} \sum_{i,k=1}^{n} t_{kj}(v)t_{il}(u) \otimes e_i \otimes e_k.$$

Thus, it remains to verify that

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u-v}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)). \tag{6.6}$$

Indeed, we have

$$[t_{ij}(u), t_{kl}(v)] = [\delta_{ij} + u^{-1}E_{ij}, \delta_{kl} + v^{-1}E_{kl}] = \frac{1}{uv}[E_{ij}, E_{kl}] = \frac{1}{uv}(\delta_{jk}E_{il} - \delta_{li}E_{kj}),$$

and

$$t_{kj}(u)t_{il}(v) - t_{kj}(u)t_{il}(v) = (\delta_{kj} + u^{-1}E_{kj})(\delta_{il} + v^{-1}E_{il}) - (\delta_{kj} + v^{-1}E_{kj})(\delta_{il} + u^{-1}E_{il}) =$$
$$= (v^{-1} - u^{-1})(\delta_{kj}E_{il} - \delta_{il}E_{kj}) = \frac{1}{uv}(u-v)(\delta_{kj}E_{il} - \delta_{il}E_{kj}),$$

and (6.6) follows. ☐

We need the following generalization of ternary relation:

Proposition 6.7. *For any $m \geq 2$ commuting indeterminates $u_1, \ldots, u_m$*

$$R(u_1, \ldots, u_m)T_1(u_1)\ldots T_m(u_m) = T_m(u_m)\ldots T_1(u_1)R(u_1, \ldots, u_m).$$

Proof. To simplify the notation denote $T_i = T_i(u_i)$. We induct on m, the base case $m = 2$ being just the ternary relation (6.5). For $m \geq 3$ note that by assumption and (6.2) we have

$$\begin{aligned} R(u_1, \dots, u_m)T_1 \dots T_m &= R_{12} \dots R_{1,m}R'(u_2, \dots, u_m)T_1T_2 \dots T_m = \\ &= R_{12} \dots R_{1,m}T_1R'(u_2, \dots, u_m)T_2 \dots T_m = \\ &= R_{12} \dots R_{1,m}T_1T_m \dots T_2R'(u_2, \dots, u_m). \end{aligned}$$

Applying ternary relations $R_{1i}T_1T_i = T_iT_1R_{1i}$ for $i = 2, \dots, m$ we get

$$\begin{aligned} R(u_1, \dots, u_m)T_1 \dots T_m &= (R_{12} \dots R_{1,m}T_1T_m \dots T_2)R'(u_2, \dots, u_m) = \\ &= R_{12} \dots R_{1,m-1}T_mT_1R_{1,m}T_{m-1} \dots T_2R'(u_2, \dots, u_m) = \\ &= T_m(R_{12} \dots R_{1,m-1}T_1T_{m-1} \dots T_2)R_{1,m}R'(u_2, \dots, u_m) = \\ &= T_m \dots T_{i+1}(R_{12} \dots R_{1,i}T_1T_i \dots T_2)R_{1,i+1} \dots R_{1,m}R'(u_2, \dots, u_m) = \\ &= T_m \dots T_2T_1R_{12} \dots R_{1,m}R'(u_2, \dots, u_m) = \\ &= T_m \dots T_2T_1R(u_1, \dots, u_m), \end{aligned}$$

which completes the inductive step. □

6.4. **Bethe subalgebras.** Define the *trace* on $(\operatorname{End} \mathbb{C}^n)^{\otimes m}$ as the linear map tr_m that acts on basis elements by

$$\operatorname{tr}_m (e_{i_1j_1} \otimes \dots \otimes e_{i_mj_m}) = \delta_{i_1j_1} \dots \delta_{i_mj_m}.$$

Observe that if we view $A \in (\operatorname{End} \mathbb{C}^n)^{\otimes}$ as an element of $\operatorname{End} (\mathbb{C}^n)^{\otimes m}$, then $\operatorname{tr}_m(A)$ is indeed the trace of the operator A acting on $(\mathbb{C}^n)^{\otimes m}$. We extend tr_m to a map $\operatorname{tr}_m : \mathcal{A} \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m} \to \mathcal{A}$ where $\mathcal{A}$ is an arbitrary non-commutative algebra:

$$\operatorname{tr}_m (a \otimes e_{i_1j_1} \otimes \dots \otimes e_{i_mj_m}) = \delta_{i_1j_1} \dots \delta_{i_mj_m}a, \; a \in \mathcal{A}.$$

We need some properties of the trace map.

Lemma 6.8 (Cyclicity property)**.** *For any $A \in \mathcal{A} \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m}$ and $B \in 1 \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes m}$ we have*

$$\operatorname{tr}_m AB = \operatorname{tr}_m BA. \tag{6.7}$$

Proof. It suffices to check the identity when $A = a \otimes e_{i_1j_1} \otimes \dots \otimes e_{i_mj_m}$ and $B = 1 \otimes e_{k_1l_1} \otimes \dots \otimes e_{k_ml_m}$. Indeed, in this case we have $\operatorname{tr}_m AB = \operatorname{tr}_m BA = a \cdot \delta_{j_1k_1}\delta_{i_1l_1} \dots \delta_{j_mk_m}\delta_{i_ml_m}$. □

Lemma 6.9. *For any positive integer $k \leq n$ and any $n \times n$ matrices $C^{(1)}, \dots, C^{(k)}$ whose entries belong to a possible non-commutative algebra we have*

$$\operatorname{tr}_n A_nC_1^{(1)} \dots C_k^{(k)} = (n-k)! \cdot \operatorname{tr}_k A_kC_1^{(1)} \dots C_k^{(k)}.$$

Proof. For each $1 \leq l \leq k$ we let $C^{(l)} = \sum_{i,j} c_{ij}^{(l)} \otimes e_{ij}$. Note that for any $k \leq m \leq n$ we have

$$A_mC_1^{(1)} \dots C_k^{(k)} = \sum_{\substack{i_1,\dots,i_m=1 \\ j_1,\dots,j_k=1}}^{n} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma)c^{(1)}_{i_{\sigma(1)}j_1} \dots c^{(k)}_{i_{\sigma(k)}j_k} \cdot e_{i_1j_1} \otimes \dots \otimes e_{i_kj_k} \otimes e_{i_{k+1}i_{\sigma(k+1)}} \otimes \dots \otimes e_{i_mi_{\sigma(m)}}.$$

Taking the traces of both sides gives

$$\operatorname{tr}_m A_mC_1^{(1)} \dots C_k^{(k)} = \sum_{i_1,\dots,i_m=1}^{n} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma)c^{(1)}_{i_{\sigma(1)}i_1} \dots c^{(k)}_{i_{\sigma(k)}i_k} \cdot \delta_{i_{k+1}i_{\sigma(k+1)}} \dots \delta_{i_mi_{\sigma(m)}}.$$

Observe that any m-tuple $(i_1, \dots, i_m) \in [n]^m$ which contains equal entries the corresponding summand in the expression above vanishes. Thus,

$$\begin{aligned} \operatorname{tr}_m A_mC_1^{(1)} \dots C_k^{(k)} &= \sum_{(i_1,\dots,i_m) \in [n]^{\underline{m}}} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma)c^{(1)}_{i_{\sigma(1)}i_1} \dots c^{(k)}_{i_{\sigma(k)}i_k} \cdot \delta_{i_{k+1}i_{\sigma(k+1)}} \dots \delta_{i_mi_{\sigma(m)}} = \\ &= \sum_{(i_1,\dots,i_m) \in [n]^{\underline{m}}} \sum_{\substack{\sigma \in \mathfrak{S}_m \\ \sigma(j)=j,\, k<j\leq m}} \operatorname{sgn}(\sigma)c^{(1)}_{i_{\sigma(1)}i_1} \dots c^{(k)}_{i_{\sigma(k)}i_k} = \\ &= \frac{(n-k)!}{(n-m)!} \sum_{(i_1,\dots,i_k) \in [n]^{\underline{k}}} \sum_{\tau \in \mathfrak{S}_k} \operatorname{sgn}(\tau)c^{(1)}_{i_{\tau(1)}i_1} \dots c^{(k)}_{i_{\tau(k)}i_k}. \end{aligned}$$

Applying this formula for $m = n$ and $m = k$ we get the required identity. □

Consider the elements $\tau_k(u)$ defined as follows:

$$\tau_k(u) = \frac{1}{k!} \operatorname{tr}_k A_kT_1(u) \dots T_k(u-k+1).$$

Proposition 6.10. *All coefficients of the series $\tau_1(u), \dots, \tau_n(u)$ commute.*

Proof. In order to prove the commutativity it suffices to verify that $\tau_k(u)\tau_l(v) = \tau_l(v)\tau_k(u)$ for all k, l. Note that this is equivalent to the equality

$$\operatorname{tr}_{k+l} A_k T_1(u) \dots T_k(u-k+1) A'_l T_{k+1}(v) \dots T_{k+l}(v-l+1) = \operatorname{tr}_{k+l} A'_l T_{k+1}(v) \dots T_{k+l}(v-l+1) A_k T_1(u) \dots T_k(u-k+1).$$

To check this equality we start with relation (6.7) in $k+l$ variables

$$R(u_1, \dots, u_{k+l}) T_1(u_1) \dots T_{k+l}(u_{k+l}) = T_{k+l}(u_{k+l}) \dots T_1(u_1) R(u_1, \dots, u_{k+l})$$

and then specialize variables $\{u_i\}$ as in (6.4)

$$u_i = u - i + 1, \ u_{k+j} = v - j + 1, \ 1 \le i \le k, 1 \le j \le l.$$

For brevity we set $T_a = T_a(u_a)$. Then, Lemma 6.5 implies

$$\widetilde{R}(u,v) A_k A'_l T_1 \dots T_k T_{k+1} \dots T_{k+l} = T_{k+l} \dots T_{k+1} T_k \dots T_1 A'_l A_k \widetilde{R}(u,v),$$

and applying generalized ternary relations for k and l parameters gives

$$\widetilde{R}(u,v) A_k T_1 \dots T_k A'_l T_{k+1} \dots T_{k+l} = A'_l T_{k+1} \dots T_{k+l} A_k T_1 \dots T_k \widetilde{R}(u,v).$$

Finally, since $\widetilde{R}(u,v)$ is invertible, we can multiply by $\widetilde{R}(u,v)^{-1}$ on th right and take the trace. Using the cyclicity property of the trace we get

$$\begin{aligned}\operatorname{tr}_{k+l} A_k T_1 \dots T_k A'_l T_{k+1} \dots T_{k+l} &= \operatorname{tr}_{k+l} \widetilde{R}(u,v) A_k T_1 \dots T_k A'_l T_{k+1} \dots T_{k+l} \widetilde{R}(u,v)^{-1} \\ &= \operatorname{tr}_{k+l} A'_l T_{k+1} \dots T_{k+l} A_k T_1 \dots T_k,\end{aligned}$$

which proves the required identity. □

Lemma 6.11. *For any complex $n \times n$ matrix C the matrix $C \cdot T(u)$ also satisfies the ternary relation* (6.5).

Corollary 6.12. *For any complex $n \times n$ matrix C all coefficients of the series*

$$\tau_k(u, C) = \frac{1}{k!} \operatorname{tr}_k A_k C_1 \dots C_k T_1(u) \dots T_k(u-k+1)$$

commute.

Proof. The ternary relation (6.5) still holds if we replace the matrix $T(u)$ with $C \cdot T(u)$. Therefore, we can repeat the proof of Proposition 6.10 to show the commutativity of $\tau_k(u, C)$. □

Corollary 6.13. *For any complex $n \times n$ matrix C all coefficients of the series*

$$\sigma_k(u, C) = \frac{1}{n!} \operatorname{tr}_n A_n T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n$$

commute.

Proof. Note that the elements $\sigma_k(u, C)$ depend polynomially on the entries of C. Hence, it suffices to check the commutativity for invertible $n \times n$ matrix C. In this case we have

$$\begin{aligned}\sigma_k(u, C) &= \frac{1}{n!} \operatorname{tr}_n A_n T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n = \\ &= \frac{1}{n!} \operatorname{tr}_n A_n C_1 \dots C_n C_1^{-1} T_1(u) \dots C_k^{-1} T_k(u-k+1) = \\ &= \frac{1}{n!} \det(C) \cdot \operatorname{tr}_n A_n C_1^{-1} T_1(u) \dots C_k^{-1} T_k(u-k+1) = \\ &= \frac{(n-k)!}{n!} \det(C) \cdot \operatorname{tr}_k A_k C_1^{-1} T_1(u) \dots C_k^{-1} T_k(u-k+1) = \\ &= \frac{k! \cdot (n-k)!}{n!} \det(C) \cdot \tau_k(u, C^{-1})\end{aligned}$$

since $A_n C_1 \dots C_n = \det(C) A_n$. Therefore, $\sigma_k(u, C)$ is a non-zero multiple of $\tau_k(u, C^{-1})$ and the statement now follows from Corollary 6.12. □

In calculations we also use alternative formulas for elements $\tau_k(u, C)$ and $\sigma_k(u, C)$.

Lemma 6.14. *In the notation of Corollaries 6.12 and 6.13 we have*

$$\tau_k(u, C) = \frac{1}{k!} \operatorname{tr}_k A_k C_1 \dots C_k T_k(u-k+1) \dots T_1(u), \tag{6.8}$$

$$\sigma_k(u, C) = \frac{1}{n!} \operatorname{tr}_n A_n T_k(u-k+1) \dots T_1(u) C_{k+1} \dots C_n. \tag{6.9}$$

Proof. From the proofs of the above corollaries we have

$$\tau_k(u,C) = \frac{1}{k!}\operatorname{tr}_k A_k C_1 T_1(u)\dots C_k T_k(u-k+1).$$

Since the matrix $C\cdot T(u)$ satisfies (6.5) and hence by (6.7) we have

$$\tau_k(u,C) = \frac{1}{k!}\operatorname{tr}_k C_k T_k(u-k+1)\dots C_1 T_1(u) A_k = \frac{1}{k!}\operatorname{tr}_k A_k C_1\dots C_k T_k(u-k+1)\dots T_1(u)$$

by the cyclicity property of the tr_k. This proves the first formula. To prove the second formula we again conside the case when C is invertible. Then, as in the proof of Corollary 6.13, we have

$$\begin{aligned}\sigma_k(u,C) &= \frac{(n-k)!}{n!}\det(C)\cdot\operatorname{tr}_k A_k C_1^{-1}T_1(u)\dots C_k^{-1}T_k(u-k+1) = \\ &= \frac{(n-k)!}{n!}\det(C)\cdot\operatorname{tr}_k A_k C_k^{-1}T_k(u-k+1)\dots C_1^{-1}T_1(u) = \\ &= \frac{1}{n!}\det(C)\cdot\operatorname{tr}_n A_n C_k^{-1}T_k(u-k+1)\dots C_1^{-1}T_1(u) = \\ &= \frac{1}{n!}\cdot\operatorname{tr}_n A_n C_1\dots C_n(C_k^{-1}\dots C_1^{-1})T_k(u-k+1)\dots T_1(u) = \\ &= \frac{1}{n!}\cdot\operatorname{tr}_n A_n T_k(u-k+1)\dots T_1(u) C_{k+1}\dots C_n\end{aligned}$$

by the first formula and the fact that $A_n C_1\dots C_n = \det(C)A_n$. This concludes the proof of the lemma. □

6.5. **Application to Capelli's identities.** Let us show how the R-matrix formalism can be used in order to prove variants of the Capelli identity.

Lemma 6.15. *For any k-tuples $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_k)$ in $[n]^k$ we have the following identity in $U(\mathfrak{gl}_n)[z]$:*

$$\operatorname{rdet}[E_{i_\alpha j_\beta} + (z+\alpha-1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k = \operatorname{cdet}[E_{i_\alpha j_\beta} + (z+k-\alpha)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.$$

In particular, these expressions are skew-symmetric in $i_1,\dots,i_k$ and $j_1,\dots,j_k$, respectively.

Proof. We start with the ternary relation for $m=k$ (see Proposition 6.7):

$$A_k T_1(u)\dots T_k(u-k+1) = T_k(u-k+1)\dots T_1(u)A_k.$$

Now let us apply both sides as operators on $(\mathbb{C}^n)^{\otimes k}$ to a vector $e_{j_1}\otimes\dots e_{j_k}$. The left-hand side gives

$$\begin{aligned}(A_k T_1(u)\dots T_k(u-k+1))(e_{j_1}\otimes\dots e_{j_k}) &= \sum_{l_1,\dots,l_k=1}^n t_{l_1 j_1}(u)\dots t_{l_k j_k}(u-k+1)\cdot A_k(e_{l_1}\otimes\dots e_{l_k}) = \\ &= \sum_{l_1,\dots,l_k=1}^n\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)t_{l_1 j_1}(u)\dots t_{l_k j_k}(u-k+1)\cdot e_{l_{\sigma(1)}}\otimes\dots\otimes e_{l_{\sigma(k)}} = \\ &= \sum_{l_1,\dots,l_k=1}^n\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)t_{l_{\sigma(1)} j_1}(u)\dots t_{l_{\sigma(k)} j_k}(u-k+1)\cdot e_{l_1}\otimes\dots e_{l_k}.\end{aligned}$$

In particular, the coefficient in front of $e_{i_1}\otimes\dots\otimes e_{i_k}$ equals

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)t_{i_{\sigma(1)}j_1}(u)\dots t_{i_{\sigma(k)}j_k}(u-k+1) &= (u^{\underline{k}})^{-1}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)\prod_{p=1,\dots,k}(E_{i_{\sigma(p)}j_p} + (u-p+1)\delta_{i_{\sigma(p)}j_p}) = \\ &= (u^{\underline{k}})^{-1}\cdot\operatorname{cdet}[E_{i_\alpha j_\beta} + (u-\alpha+1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.\end{aligned}$$

On the other hand,

$$\begin{aligned}(T_k(u-k+1)\dots T_1(u)A_k)(e_{j_1}\otimes\dots\otimes e_{j_k}) &= \sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)(T_k(u-k+1)\dots T_1(u))(e_{j_{\sigma(1)}}\otimes\dots\otimes e_{j_{\sigma(k)}}) = \\ &= \sum_{\sigma\in\mathfrak{S}_k}\sum_{l_1,\dots,l_k=1}^n\operatorname{sgn}(\sigma)t_{l_1 i_{\sigma(1)}}(u-k+1)\dots t_{l_k i_{\sigma(k)}}(u)\cdot e_{l_1}\otimes\dots\otimes e_{l_k}.\end{aligned}$$

Hence, the coefficient in front of $e_{i_1}\otimes\dots\otimes e_{i_k}$ equals

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)t_{i_1 j_{\sigma(1)}}(u-k+1)\dots t_{i_k j_{\sigma(k)}}(u) &= (u^{\underline{k}})^{-1}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)\prod_{p=1,\dots,k}(E_{i_p j_{\sigma(p)}} + (u-k+p)\delta_{i_p j_{\sigma(p)}}) = \\ &= (u^{\underline{k}})^{-1}\cdot\operatorname{rdet}[E_{i_\alpha j_\beta} + (u-k+\alpha)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.\end{aligned}$$

Combining everything together, we obtain

$$\operatorname{rdet}[E_{i_\alpha j_\beta} + (u-k+\alpha)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k = \operatorname{cdet}[E_{i_\alpha j_\beta} + (u-\alpha+1)\delta_{i_\alpha j_\beta}]_{\alpha,\beta=1}^k.$$

Substituting $u = z + k - 1$ finishes the proof of the lemma. □

Remark 6.6. In particular, by plugging $z = 0$ into the lemma we obtain another proof of Corollary 5.6.

7. Commutativity of the big algebra

The aim of this section is to prove the following crucial result.

Theorem 7.1. *The algebra $\mathscr{B}(\mathcal{P}(n,r))$ is a commutative.*

Remark 7.1. Known proofs of this fact involve quite non-trivial constructions such as *Feigin-Frenkel center*, *Segal-Sugawara vectors* and *opers*. The main advantage of our approach is that it is essentially elementary and relies on direct calculations and known facts about the representation theory of classical Lie algebras.

Proof. In view of Corollary 3.9 it suffices to check that the operators $F_{p,q}$ commute. By (3.7) and (5.3)

$$F_{p,q}(Y) = \sum_{\substack{I_1,J_1\in\binom{[n]}{p}\\ I_2,J_2\in\binom{[n]}{q}\\ I_1\sqcup I_2=J_1\sqcup J_2}} \operatorname{sgn}\begin{pmatrix} I_1 & I_2\\ J_1 & J_2\end{pmatrix} \det Y_{I_1J_1}\cdot L(\Pi_{J_2I_2}).$$

Note that in view of Proposition 3.4, we only need to check the commutativity for $Y \in \mathfrak{h}$, i.e. for diagonal matrices. For $Y = \operatorname{diag}(z_1,\dots,z_n)$ we have

$$F_{p,q}(Y) = \sum_{\substack{I\in\binom{[n]}{p}\\ J\in\binom{[n]}{q}\\ I\cap J=\varnothing}} \det Y_{II}\cdot L(\Pi_{JJ}) = \sum_{\substack{I\in\binom{[n]}{p}\\ J\in\binom{[n]}{q}\\ I\cap J=\varnothing}} \prod_{i\in I} z_i\cdot L(\Pi_{JJ}).$$

Notice that by Lemma 6.14 for the diagonal matrix $C = \operatorname{diag}(z_1,\dots,z_n)$ we have

$$\begin{aligned} n!\cdot\sigma_k(u,C) &= \operatorname{tr}_n A_nT_k(u-k+1)\dots T_1(u)C_{k+1}\dots C_n = \\ &= \sum_{i_1,\dots,i_n}\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)t_{i_1i_{\sigma(1)}}(u-k+1)\dots t_{i_ki_{\sigma(k)}}(u)\prod_{j=k+1}^{n}\delta_{i_ji_{\sigma(j)}}z_{i_j} = \\ &= \sum_{(i_1,\dots,i_k)\in[n]^{\underline{k}}}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)t_{i_1i_{\sigma(1)}}(u-k+1)\dots t_{i_ki_{\sigma(k)}}(u)\cdot\prod_{i\in I^c}z_i = \\ &= k!\sum_{i_1<\dots<i_k}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)t_{i_1i_{\sigma(1)}}(u-k+1)\dots t_{i_ki_{\sigma(k)}}(u)\cdot\prod_{i\in I^c}z_i = \\ &= k!\cdot(u(u-1)\dots(u-k+1))^{-1}\sum_{I\in\binom{[n]}{k}}\Pi_{II}(k-1-u)\cdot\prod_{i\in I^c}z_i, \end{aligned}$$

where by I^c we denote the complement of $I = \{i_1,\dots,i_k\}$ in $[n]$. Here we used that the expression

$$\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}(\sigma)t_{i_1i_{\sigma(1)}}(u-k+1)\dots t_{i_ki_{\sigma(k)}}(u) = (u^{\underline{k}})^{-1}\cdot\operatorname{rdet}[E_{i_\alpha i_\beta} + (u-k+\alpha)\delta_{i_\alpha i_\beta}]_{\alpha,\beta=1}^{k}$$

is skew-symmetric in $i_1,\dots,i_k$ (see Lemma 6.15).

Thus,

$$\begin{aligned} n!\cdot(-1)^{n-p}\binom{u}{n-p}\cdot\sigma_{n-p}(n-p-1-u,C) &= \sum_{I\in\binom{[n]}{n-p}}\Pi_{II}(u)\cdot\prod_{i\in I^c}z_i = \\ &= \sum_{I\in\binom{[n]}{n-p}}\sum_{\ell=0}^{n-p}(-1)^{n-p-\ell}u^{\underline{n-p-\ell}}\sum_{K\in\binom{I}{\ell}}\Pi_{KK}\cdot\prod_{i\in I^c}z_i = \sum_{\ell=0}^{n-p}(-1)^{n-p-\ell}u^{\underline{n-p-\ell}}\sum_{\substack{J\in\binom{n}{p}\\ K\in\binom{[n]}{\ell}\\ J\cap K=\varnothing}}\Pi_{KK}\prod_{j\in J}z_j. \end{aligned}$$

Applying the map L, we obtain

$$n!\cdot(-1)^{n-p}\binom{u}{n-p}\cdot L(\sigma_{n-p}(n-p-1-u,C)) = \sum_{\ell=0}^{n-p}(-1)^{n-p-\ell}u^{\underline{n-p-\ell}}\cdot F_{p,\ell}(Y).$$

Now Corollary 6.13 implies that for any $0\le p_1,p_2\le n$ we have the identity

$$\left[\sum_{\ell_1=0}^{n-p_1}(-1)^{n-p_1-\ell_1}u^{\underline{n-p_1-\ell_1}}\cdot F_{p_1,\ell_1}(Y),\ \sum_{\ell_2=0}^{n-p_2}(-1)^{n-p_2-\ell_2}v^{\underline{n-p_2-\ell_2}}\cdot F_{p_2,\ell_2}(Y)\right] = 0.$$

Finally, polynomials $\{u^{\underline{k}}v^{\underline{l}}\}_{k,l\geq 0}$ are linearly independent and hence, $[F_{p_1,q_1}(Y), F_{p_2,q_2}(Y)] = 0$ for all p_1, p_2, q_1, q_2. □

Remark 7.2. In the course of the proof we actually showed that (see **??**)

$$n! \cdot (-1)^{n-p} \binom{u}{n-p} \cdot L(\sigma_{n-p}(n-p-1-u, C)) = \sum_{\ell=0}^{n-p} (-1)^{n-p-\ell} u^{\underline{n-p-\ell}} \cdot F_{p,\ell}(Y),$$

$$L(\sigma_{n-p}(v, Y)) = \sum_{\ell=0}^{n-p} (v(v-1)\dots(v-\ell+1))^{-1} \cdot F_{p,\ell}(Y).$$

In fact, the elements of Bethe algebra as functions in $C \in \operatorname{End}(\mathbb{C}^n)$ can be regarded as elements of Kirillov algebra in the sense of the following lemma.

Lemma 7.2. *The coefficients of the power series $C \mapsto L(\sigma_k(u, C^T))$ belong to $\mathscr{C}(\mathcal{P}(n,r))$.*

Proof. Observe that for any $g \in \mathrm{GL}_n$ the element A_n commutes with $g_1 g_2 \dots g_n$ in $\operatorname{End}(\mathbb{C}^n)^{\otimes n}$. Hence, by the cyclicity property of the trace,

$$\begin{aligned} n! \cdot \sigma_k(u, \operatorname{Ad}(g)(C)) &= n! \cdot \sigma_k(u, gCg^{-1}) = \operatorname{tr}_n A_n T_1(u) \dots T_k(u-k+1) g_{k+1} C_{k+1} g_{k+1}^{-1} \dots g_n C_n g_n^{-1} = \\ &= \operatorname{tr}_n A_n g_{k+1} \dots g_n T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n g_{k+1}^{-1} \dots g_n^{-1} = \\ &= \operatorname{tr}_n A_n (g_1 \dots g_n)(g_1^{-1} \dots g_k^{-1}) T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n g_{k+1}^{-1} \dots g_n^{-1} = \\ &= \operatorname{tr}_n A_n g_1^{-1} \dots g_k^{-1} T_1(u) \dots T_k(u-k+1) C_{k+1} \dots C_n g_1 \dots g_k = \\ &= \operatorname{tr}_n A_n g_1^{-1} T_1(u) g_1 \dots g_k^{-1} T_k(u-k+1) g_k C_{k+1} \dots C_n. \end{aligned}$$

On the other hand, for any $h \in \mathrm{GL}_n$ we have

$$\begin{aligned} (\operatorname{id} \otimes h) T(v) (\operatorname{id} \otimes h)^{-1} &= \sum_{i,j=1}^n t_{ij}(v) \otimes \operatorname{Ad}(h)(e_{ij}) = \sum_{k,l=1}^n \sum_{i,j=1}^n [\operatorname{Ad}(h)(e_{ij})]_{kl} \cdot (\delta_{ij} + v^{-1} E_{ij}) \otimes e_{kl} = \\ &= \sum_{k,l=1}^n \left(\sum_{i,j=1}^n [\operatorname{Ad}(h)(e_{ij})]_{kl} \cdot \delta_{ij} \right) \otimes e_{kl} + \sum_{k,l=1}^n v^{-1} \left(\sum_{i,j=1}^n [\operatorname{Ad}(h)(e_{ij})]_{kl} \cdot E_{ij} \right) \otimes e_{kl} = \\ &= \sum_{k,l=1}^n (\delta_{kl} + v^{-1} \operatorname{Ad}(h^T)(E_{kl})) \otimes e_{kl} = \sum_{k,l=1}^n \operatorname{Ad}(h^T)(t_{kl}(v)) \otimes e_{kl}, \end{aligned}$$

where the last equality follows from

$$\begin{aligned} \sum_{i,j=1}^n [\operatorname{Ad}(h)(e_{ij})]_{kl} \cdot E_{ij} &= \sum_{i,j=1}^n \operatorname{tr}(\operatorname{Ad}(h)(e_{ij})\, e_{lk}) \cdot E_{ij} = \sum_{i,j=1}^n \operatorname{tr}(e_{ij} \operatorname{Ad}(h^{-1})(e_{lk})) \cdot E_{ij} = \\ &= \sum_{i,j=1}^n \operatorname{tr}(\operatorname{Ad}(h^T)(e_{kl})\, e_{ji}) \cdot E_{ij} = \sum_{i,j=1}^n [\operatorname{Ad}(h^T)(e_{kl})]_{ij} \cdot E_{ij} = \operatorname{Ad}(h^T)(E_{kl}). \end{aligned}$$

Therefore, $(\operatorname{id} \otimes h) T(v) (\operatorname{id} \otimes h)^{-1} = (\operatorname{Ad}(h^T) \otimes \operatorname{id})(T(v))$. It follows that for any $g \in \mathrm{GL}_n$ we have

$$\sigma_k(u, \operatorname{Ad}(g)(C)) = \operatorname{Ad}\left((g^T)^{-1}\right)(\sigma_k(u, C)).$$

If we denote $\Phi(u, C) = L(\sigma_k(u, C^T))$, then for any $g \in \mathrm{GL}_n$ we obtain

$$\Phi(u, \operatorname{Ad}(g)(C)) = L(\operatorname{Ad}(g)(\sigma_k(u, C))) = \widetilde{L}(g) \Phi(u, C) \widetilde{L}(g)^{-1},$$

which is precisely the condition (3.2). Hence, all coefficients of the u-power series $\Phi(u, C) = L(\sigma_k(u, C^T))$ belong to $\mathscr{C}(\mathcal{P}(n,r))$. □

8. Big algebra in the multiplicity free case

The goal of this section is to show that big algebra coincides with the medium algebra in the multiplicity free case.

Proposition 8.1. *Assume that $r = 1$. Then, the big algebra $\mathscr{B}(\mathcal{P}(n,r))$ coincides with the medium algebra $\mathscr{M}(\mathcal{P}(n,r))$.*

Proof. Indeed, we just note that in the case $r = 1$ all operators $F_{p,q}$ with $q > 1$ are zero. Therefore, both algebras $\mathscr{M}(\mathcal{P}(n,r))$ and $\mathscr{B}(\mathcal{P}(n,r))$ are generated by operators $\{F_{p,0}, F_{p,1} : p \leq n\}$. Hence, $\mathscr{M}(\mathcal{P}(n,r)) = \mathscr{B}(\mathcal{P}(n,r))$. □

Remark 8.1. T. Hausel proves that medium algebra $\mathscr{M}(V)$ is the center of Kirillov algebra $\mathscr{C}(V)$ in the case of irreducible V (see [2, Theorem 1.1]). On the other hand, N. Rozhkovskaya shows that $\mathscr{C}(V)$ is commutative if and only if V is weight multiplicity free (see [11, Theorem 4.1]). Since in case $r = 1$ the ring $\mathcal{P}(r, n)$ is isomorphic to $S(\mathbb{C}^n)$ as a $\mathfrak{gl}_n$-representation, its decomposition into irreducibles is just $\bigoplus_{k\geq 0} S^k(\mathbb{C}^n)$. Note that all these summands are weight mutliplicity-free. Thus, taking into account Proposition 3.7, one can view Proposition 8.1 as a combination of the results by Hausel and Rozhkovskaya.

9. Miscellaneous facts

9.1. **About restriction to Cartan subalgebra.** Note that Kirillov algebra $\mathscr{C}(V)$ is not in general commutative. However, if V is weight multiplicity free, then it is the case. This is an immediate consequence of the following fact.

Proposition 9.1. *Let V be a weight mtliplicity free representation of $\mathfrak{g} = \mathfrak{gl}_n$. Then, for any $A \in \mathscr{C}(V)$ and any $x \in \mathfrak{h}$ the operator $A(x) \in \operatorname{End} V$ is diagonal in the weight basis of V.*

Remark 9.1. Note that since V is weight multiplicity free, the elements of weight basis of V are determined uniquely up to multiplication by a non-zero scalar.

9.2. **Formula for D-operator.** Let m be a positive integer. Consider $V = V(m\varpi_1) = S^m(\mathbb{C}^n)$, i.e. the m-th symmetric power of the standard representation of $\mathfrak{gl}_n$. Then, V is weight multiplicity free and its weights are just n-tuples $\mu = (\mu_1, \dots, \mu_n)$, where $\mu_i \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=1}^n \mu_i = m$.

Our aim is to obtain a description of big algebra $\mathscr{B}(S^m(\mathbb{C}^n))$ in terms of generators and relations. Besides that, we would like to get a simple formula for D-operator.

Consider the ring $DI_n = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{\mathfrak{S}_n}$, where $\mathfrak{S}_n$ acts diagonally on variables x_i, y_i. It is known [13, Ch. II, §A.3] that this ring is generated by the so-called *polarized power sums*

$$p_{r,s} = \sum_{i=1}^n x_i^r y_i^s.$$

Note that these generators are not algebraically independent unlike the case.

Let DI_n^1 be the subring of DI_n generated by $p_{s,0}$ and $p_{s,1}$ for all $s \geq 0$.

Proposition 9.2. *Big algebra $\mathscr{B}(S^m(\mathbb{C}^n))$ is isomorphic to a certain quotient of the ring DI_n^1. Namely, $\mathscr{B}(S^m(\mathbb{C}^n))$ is isomorphic to the image of DI_n^1 under the map*

$$\bigoplus_{\mu \vdash m\varpi_1} \mathrm{ev}_\mu,$$

where the ring homomorphism $\mathrm{ev}_\mu \colon DI_n^1 \to S(\mathfrak{h}^*) \otimes \operatorname{End} V_\mu(m\varpi_1)$ sends $p_{r,s}$ with $0 \leq s \leq 1$ to $\sum_{i=1}^n \mu_i^s t_i^r$. Here we identify $S(\mathfrak{h}^*)$ with $\mathbb{C}[t_1, \dots, t_n]$ and also view $\mathscr{B}(S^m(\mathbb{C}^n))$ as a subalgebra inside $S(\mathfrak{h}^*) \otimes \operatorname{End}_{\mathfrak{h}} S^m(\mathbb{C}^n)$.

Denote by Λ the ring of symmetric functions in infinitely many variables. For $f \in \Lambda$ we use the notation $f[A]$ for the *plethystic substitution*, where A is a certain expression in x_i, y_i. We treat x_i as variables and y_i as constants.

Proposition 9.3. *For any $f, g \in \Lambda$ define*

$$\widehat{\mathbf{D}}(f[y_1 + \ldots + y_n]g[x_1y_1 + \ldots + x_ny_n]) = \left(y_1\frac{\partial}{\partial x_1} + \ldots + y_n\frac{\partial}{\partial x_n}\right)(f[x_1 + \ldots + x_n]g[x_1y_1 + \ldots + x_ny_n]) + \\ + f[y_1 + \ldots + y_n] \cdot \sum_{i\neq j} y_j(y_i + 1)\frac{g[x_1y_1 + \ldots + x_ny_n + (x_i - x_j)] - g[x_1y_1 + \ldots + x_ny_n)]}{x_i - x_j}.$$

The map $\widehat{\mathbf{D}}$ induces the D-operator on $\mathscr{B}(S^m(\mathbb{C}^n))$.

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