CLASSICAL AND QUANTUM FAMILY ALGEBRAS OF THE FUNDAMENTAL REPRESENTATIONS OF $\mathfrak{gl}_n$

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Abstract. In this note we study the so called family algebras of the fundamental irreducible representations of $\mathfrak{gl}_n$. We describe certain distiguished elements of these algebras, called M-elements.

1. Introduction

1.1. **Classical family algebras.** Alexander Kirillov in his papers [4] and [5] introduced a new class of algebras $\mathcal{C}_\mu(\mathfrak{g})$ (the so called *classical family algebras*) which are related to irreducible representations of Lie algebras. These algebras proved to be useful in different problems of representation theory and mathematical physics.

Let G be a reductive Lie group and let $\mathfrak{g}$ be its Lie algebra. Assume that $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is an invariant non-degenerate bilinear form on $\mathfrak{g}$. For example, if $\mathfrak{g}$ is semisimple, then one can choose B to be the Killing form.

Let (π_μ, V_μ) be the irreducible representation of $\mathfrak{g}$ with the highest weight μ of dimension $d(\mu)=\dim V_\mu$. Observe that $\operatorname{End} V_\mu$ possesses a natural structure of a G-module. Indeed, the acion of G on $\operatorname{End} V_\mu$ is given by the formula

$$g\cdot A=\pi_\mu(g)A\pi_\mu(g)^{-1}.$$

Besides that, the adjoint action Ad of the group G on the corresponding Lie algebra $\mathfrak{g}$ gives rise to the G-action on $S(\mathfrak{g})$. Thus, $\operatorname{End} V_\mu\otimes S(\mathfrak{g})$ is also a G-module. Finally, define the algebra $\mathcal{C}_\mu$ as

$$\mathcal{C}_\mu(\mathfrak{g})=(\operatorname{End} V_\mu\otimes S(\mathfrak{g}))^G. \tag{1.1}$$

The algebra $C_\mu(\mathfrak{g})$ is called the *clasical family algebra* (see [4, 5]).

Since B is an invariant non-degenerate bilinear form on $\mathfrak{g}$, it produces a canonical isomorphism of G-modules $\mathfrak{g}$ and $\mathfrak{g}^*$. Hence, there is a corresponding isomorphism of G-modules $S(\mathfrak{g})\simeq S(\mathfrak{g}^*)$. Thus, one can think of $\mathcal{C}_\mu(\mathfrak{g})$ as of G-equivariant polynomial maps from $\mathfrak{g}$ to $\operatorname{End} V_\mu$. In other words, one can regard $\mathcal{C}_\mu(\mathfrak{g})$ as a subset of $d(\mu)\times d(\mu)$ matrices $A=\{A_{ij}\}_{i,j=1}^{d(\mu)}$, whose entries A_{ij} are polynomial functions on $\mathfrak{g}$, and which satisfy the equality

$$\pi_\mu(g)A\pi_\mu(g)^{-1}=A\circ\operatorname{Ad}(g) \text{ for all } g\in G. \tag{1.2}$$

Here $A\circ\operatorname{Ad}(g)$ is a $d(\mu)\times d(\mu)$ matrix whose (i,j)-entry equals $A_{ij}\circ\operatorname{Ad}(g)$.

1.2. **Quantum family algebras.** Recall that the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ possess a natural G-action. Thus, one can consider the quantum analogues of classical family algebras by taking the universal enveloping algebra $U(\mathfrak{g})$ instead of symmetric algebra $S(\mathfrak{g})$. In other words, we define the algebra $\mathcal{Q}_\mu(\mathfrak{g})$ as (see [5])

$$\mathcal{Q}_\mu(\mathfrak{g})=(\operatorname{End} V_\mu\otimes U(\mathfrak{g}))^G. \tag{1.3}$$

We call $\mathcal{Q}_\mu(\mathfrak{g})$ the *quantum family algebra*. Similarly, $\mathcal{Q}_\mu(\mathfrak{g})$ can be described as an algebra consisting of $d(\mu)\times d(\mu)$ matrices $\{A_{ij}\}$, whose entries belong to $U(\mathfrak{g})$, and which satisfy the identity

$$\pi_\mu(g)^{-1}A\pi_\mu(g)=\operatorname{Ad}(g)(A). \tag{1.4}$$

Here $\operatorname{Ad}(g)$ is the entry-wise adjoint action of G on $\operatorname{Mat}(d(\mu)\times d(\mu), U(\mathfrak{g}))$.

Remark 1.1. Note that condition (1.4) in the quantum case differs from (1.2) in the classical case. The reason is that the elements $\mathcal{C}_\mu(\mathfrak{g})$ can be identified with certain matrix-valued polynomial maps on $\mathfrak{g}$, whereas there is no similar description of $\mathcal{Q}_\mu(\mathfrak{g})$.

We also refer to family algebras, both classical and quantum, as to *Kirillov algebras*.

1.3. **M-elements and their characterstic identities.** Here we recall several constructions from the theory of family algebras (see [4] and [5]).

Date: December 28, 2022.

1.3.1. *Classical case.* Suppose that $X_1, \dots, X_m$ and $X^1, \dots, X^m$ are dual bases of $\mathfrak{g}$ with respect to $B(\cdot,\cdot)$. For any $A \in \mathcal{C}_\mu(\mathfrak{g})$ define

$$D(A) = \sum_{i=1}^{m} \pi_\mu(X_i) \cdot \frac{\partial A}{\partial X_i}.$$

One can check that this definition is actually independent of the choice of basis $X_1, \dots, X_m$.

Remark 1.2. Here $\frac{\partial A}{\partial X_i}$ means the derivative of A (it is a polynomial matrix-valued function on $\mathfrak{g}$) in the direction of $X_i \in \mathfrak{g}$.

The following fact was proved in [13, Proposition 5.2] (see also [5, Theorem M]).

Proposition 1.1. *D is a linear operator acting on $\mathcal{C}_\mu(\mathfrak{g})$.*

Note that the algebra of $\mathfrak{g}$-invariant polynomials on $\mathfrak{g}$ naturally embeds in $\mathcal{C}_\mu(\mathfrak{g})$ as subalgebra of scalar matrices. Thus, the proposition above implies that for any $\mathfrak{g}$-invariant polynomial $P \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$, the element $D(P)$ belongs to $\mathcal{C}_\mu(\mathfrak{g})$. We denote $M_P = D(P)$ and call M_P as *M-element* that corresponds to $P \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$.

Example 1.1. One important special case of this construction is when $P = C = \sum_i X_i X^i \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ is a quadratic invariant polynomial, i.e. the *quadratic Casimir element.* Then we have

$$M_C = \sum_{i=1}^{n} \pi_\mu(X_i) \cdot X^i.$$

For example, consider the case when $\mathfrak{g} = \mathfrak{gl}_n$ and $\mu = \omega_1$ (see Section 2 for the notations), i.e. when π_μ is the standard vector representation of $\mathfrak{gl}_n$. We have $C = \sum_{i,j=1}^n x_{ij}x_{ji}$ and hence

$$M_C = \begin{pmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ \dots & \dots & \dots & \dots \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{pmatrix}.$$

1.3.2. *Quantum case.* In the quantum case we do not have differentiation, but we can use the comultiplication structure on $U(\mathfrak{g})$.

Let $\Delta\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \times U(\mathfrak{g})$ be the algebra homomorphism defined on $\mathfrak{g}$ as

$$\Delta(X) = X \otimes 1 + 1 \otimes X \text{ for any } X \in \mathfrak{g}.$$

Following Kirillov we define the homomorphism $\delta\colon U(\mathfrak{g}) \to \mathrm{Mat}(d(\mu) \times d(\mu), U(\mathfrak{g}))$ as the composition $\delta = (\pi_\mu \otimes \mathrm{id}) \circ \Delta$. In particular, for $X \in \mathfrak{g}$ we have

$$\delta(X) = \pi_\lambda(X) \otimes 1 + 1 \otimes X.$$

Denote by $Z(U(\mathfrak{g}))$ the center of $U(\mathfrak{g})$. Now for any $A \in Z(U(\mathfrak{g}))$ define the element M_A as

$$M_A = \frac{1}{2}\big(\delta(A) - \pi_\mu(A) \otimes 1 - 1 \otimes A\big).$$

Since the map δ is G-equivariant, the element M_A belongs to $\mathcal{Q}_\mu(A)$. We proved the following fact.

Proposition 1.2. *For any $A \in Z(U(\mathfrak{g}))$ the element M_A belongs to $\mathcal{Q}_\mu(\mathfrak{g})$.*

As in the classical case, we also call M_A as *M-element* which corresponds to $A \in Z(U(\mathfrak{g}))$.

Example 1.2. Similar to the classical case assume that $A \in Z(U(\mathfrak{g}))$ is the quadratic Casimir element, i.e. $C = \sum_i X_i X^i \in Z(U(\mathfrak{g}))$. Then, we have

$$M_C = \sum_{i=1}^{m} \pi_\mu(X_i) \otimes X^i.$$

Remark 1.3. Notice that this formula coincides with the corresponding formula in the classical case if we use the identification of $\mathcal{C}_\mu(\mathfrak{g})$ with algebra of matrix-valued polynomial maps on $\mathfrak{g}$.

As in the first case, assume that $\mathfrak{g} = \mathfrak{gl}_n$ and $\mu = \omega_1$. Then, we have $C = \sum_{i,j=1}^n E_{ij}E_{ji}$ and

$$M_C = \begin{pmatrix} E_{11} & E_{21} & \dots & E_{n1} \\ E_{12} & E_{22} & \dots & E_{n2} \\ \dots & \dots & \dots & \dots \\ E_{1n} & E_{2n} & \dots & E_{nn} \end{pmatrix}.$$

1.3.3. *Characteristic identities for M-elements.* It turns out that M-elements satisfy certain identities of Cayley-Hamilton type. Namely, the following statement holds.

Proposition 1.3. *(a) For any $P \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ there is a polynomial of degree $d(\mu)$ with coefficients in $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ which annihilates $M_P \in \mathcal{C}_\mu(\mathfrak{g})$.*

(b) For any $A \in Z(U(\mathfrak{g}))$ there is a polynomial of degree $d(\mu)$ with coefficients in $Z(U(\mathfrak{g}))$ which annihilates $M_A \in \mathcal{Q}_\mu(\mathfrak{g})$.

Observe that this fact is trivial in the classical case because we can simply take the characteristic polynomial of M_P. However, in the quantum this is far from trivial. Moreover, one do not expect the existence of Cayley-Hamilton type identity for an arbitrary matrix with non-commuting entries.

1.4. **Connections with algebraic geometry.** It is known that the classical family algebra $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ which corresponds to the k-th fundamental representation of $\mathfrak{gl}_n$ is isomorphic to the GL_n-equivariant cohomology ring of $\mathrm{Gr}(k, n)$. Moreover, the similar fact holds for many other classical family algebras, e.g. for those corresponding to minuscule (or weight multiplicity free) representations. This relation between classical family algebras of minuscule representations and equivariant cohomology was discussed in [11, Section 6].

The quantum family algebras can be regarded as quantizations of classical ones. However, it is unclear if there is any connection between quantum family algebras and, for example, quantum cohomology of algebraic varieties.

1.5. **Results.** In this paper we are studying Kirillov algebras (both classical and quantum) for the special case $\mathfrak{g} = \mathfrak{gl}_n$ and $\mu = \omega_k$. In other words, we study algebras that correspond to the fundamental representations of $\mathfrak{gl}_n$, i.e. to the exterior powers of the standard representation of $\mathfrak{gl}_n$. These representations are minuscule, i.e. they are weight multiplicity free (see [11]). It is known that for such representations the corresponding family algebras are commutative (see [4, Theorem 1], [5, Theorem S], [10, Propositions 1 and 2] for more details).

1.5.1. *Structure of $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$.* We describe the algebra $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ in terms of generators and relations and also as a certain subalgebra of the polynomial algebra. Namely, we prove that $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ is isomorphic to the $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$-invariant part of $\mathbb{C}[t_1, \dots, t_k, t_{k+1}, \dots, t_n]$. The corresponding isomorphism is closely related to the "diagonalization homomorphism" described in Subsection 3.4.

1.5.2. *D-operator in $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$.* For any distinct elements i and j of the set $\{1, 2, \dots, n\}$ denote by $s_{ij} \in \mathfrak{S}_n$ the transposition of elements i and j. Here $\mathfrak{S}_n$ is the n-th symmetric group, i.e. the group of permutations of the set $\{1, 2, \dots, n\}$. Then, one can define the following linear operator (the divided difference) on the algebra $\mathbb{C}[t_1, \dots, t_n]$:

$$\partial_{ij}(f) = \frac{f - s_{ij} f}{t_i - t_j} \text{ for all } f \in \mathbb{C}[t_1, \dots, t_n].$$

Here we use the following notation: for any $\sigma \in \mathfrak{S}_n$ we denote $(\sigma f)(t_1, \dots, t_n) = f(t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(n)})$.

Since Kirillov algebra $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ is isomorphic to $\mathbb{C}[t_1, \dots, t_k, t_{k+1}, \dots, t_n]^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}$ we can define the action of D-operator on the latter. It turns out that one has the following explicit formula:

$$(Df)(t_1, \dots, t_n) = \left(\sum_{i=1}^{k} \frac{\partial}{\partial t_i} \right)(f) + \sum_{\substack{1 \le i \le k \\ k+1 \le j \le n}} \partial_{ij}(f)$$

We prove this formula in Subsection 3.5. Our proof uses direct computations, explicit formulas of generators and the diagonalization homomorphism.

1.5.3. *Cayley-Hamilton identities for Casimir M-operator in $\mathcal{Q}_{\omega_k}(\mathfrak{gl}_n)$.* We find analogues of Cayley-Hamilton idenities for Casimir M-operator in quantum family algebra. These can be seen as generalizations of the well-known Capelli's identities (see also [14] and references therein).

1.6. **Contents.** The present paper is organized as follows.

In Section 2 we fix the notation, revise the relevant information about reductive Lie algebras and their universal enveloping algebras. We also discuss the fundamental representations of $\mathfrak{gl}_n$.

In Section 3 we discuss the classical case. We describe $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ in terms of generators and relations and prove the explicit formula for D-operator We use this formula to obtain relations between M-elements of invariant polynomials in $\mathbb{C}[\mathfrak{gl}_n]$.

In Section 4 we discuss the quantum case. We discuss general algorithm for finding the characteristic identity for M-elements and obtain the explicit formulas in the case of quadratic Casimir element.

Finally, in Section 5 we discuss some open questions about classical and quantum family algebras.

1.7. **Acknowledgements.** This work was done during the rotation in Hausel research group at Institute of Science and Technology Austria (ISTA) under the supervision of Tamás Hausel.

2. Preliminaries for the case $\mathfrak{g} = \mathfrak{gl}_n$ and $\mu = \omega_k$

In this section we fix the notations and state several facts about the fundamental representations π_{ω_k} of the Lie algebra $\mathfrak{gl}_n$.

2.1. **General facts about complex reductive Lie algebras.** Recall that for any complex reductive Lie algebra one has the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha, \tag{2.1}$$

where R is the associated root system of $\mathfrak{g}$. Denote by R_+ (R_-) the set of positive (negative) roots in R. Then, we get the so called triangular decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{2.2}$$

where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n}_\pm = \sum_{\alpha\in R_\pm} \mathfrak{g}_\alpha$ are nilpotent subalgebras which correspond to positive and negative root spaces, respectively. This and the Poincare-Birkhoff-Witt theorem yield the following decomposition of $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \mathfrak{n}_- U(\mathfrak{g}) \oplus U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}_+. \tag{2.3}$$

In particular, one can define the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$. Since $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$ the universal enveloping algebra $U(\mathfrak{h})$ is isomorphic to the symmetric algebra $S(\mathfrak{h})$. Thus, one can define a map $\Gamma\colon U(\mathfrak{g}) \to S(\mathfrak{h})$. This map is called the *Harish-Chandra homomorphism.*

Let ρ be the half-sum of all positive roots, i.e.

$$\rho = \frac{1}{2}\sum_{\alpha\in R_+} \alpha. \tag{2.4}$$

Recall that the twisted Weyl group $\widetilde{W}$ is the Weyl group W of the root system R conjugated by the translation by ρ in $\mathfrak{h}^*$. In other words,

$$\widetilde{W} = \{\tau_\rho^{-1} \circ w \circ \tau_\rho : w \in W\},$$

where $\tau_\rho\colon \mathfrak{h}^* \to \mathfrak{h}^*$ is the translation by ρ, i.e. $\tau_\rho(\lambda) = \lambda + \rho$.

The map Γ is closely related to the center $Z(U(\mathfrak{g}))$ because of the following fact.

Theorem 2.1. *The restriction of* Γ *to* $Z(U(\mathfrak{g}))$ *is an injective algebra homomorphism, whose image consists of* $\widetilde{W}$*-invariant part of* $U(\mathfrak{h}) \simeq S(\mathfrak{h})$*.*

For any $\lambda \in \mathfrak{h}^*$ we denote by $\chi_\lambda\colon Z(U(\mathfrak{g})) \to \mathbb{C}$ the central character on $Z(U(\mathfrak{g}))$ that corresponds to a weight $\lambda \in \mathfrak{h}^*$. Then, the Harish-Chandra map Γ satisfies the following property:

$$\Gamma(A)(\lambda) = \chi_\lambda(A) \tag{2.5}$$

for any $A \in Z(U(\mathfrak{g}))$ and $\lambda \in \mathfrak{h}^*$.

We refer the reader to [7, Chapter 7, 7.4] and [12, Chapter VI, 23.3] for more information about the Harish-Chandra isomorphism.

2.2. **Triangular decomposition of $\mathfrak{gl}_n$.** We apply facts stated in Subsection 2.1 to the case $\mathfrak{g} = \mathfrak{gl}_n$.

Denote by $\{E_{ij}\}_{i,j=1}^n$ the standard basis of $\mathfrak{gl}_n$ consisting of matrix units. In particular, elements E_{ij} satisfy the relations

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}. \tag{2.6}$$

We choose Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{gl}_n$ to be the subalgebra of diagonal matrices, i.e.

$$\mathfrak{h} = \operatorname{span}\{E_{ii} : 1 \le i \le n\}$$

and choose positive roots in such a way that

$$\mathfrak{n}_- = \operatorname{span}\{E_{ij} : i > j\},\ \mathfrak{n}_+ = \operatorname{span}\{E_{ij} : i < j\}. \tag{2.7}$$

It is not difficult to show that in this case $\rho \in \mathfrak{h}^*$ acts on $\mathfrak{h}$ as

$$\rho\left(\sum_{i=1}^n h_i E_{ii}\right) = \sum_{i=1}^n \frac{n+1-2i}{2} \cdot h_i. \tag{2.8}$$

The Weyl group W in this case is isomorphic to the symmetric group $\mathfrak{S}_n$.

2.3. **Invariant form on $\mathfrak{gl}_n$.** In case $\mathfrak{g} = \mathfrak{gl}_n$ we choose form B to be $B(X,Y) = \operatorname{tr}(XY)$. It is clear that B is an invariant non-degenerate bilinear form on $\mathfrak{g} = \mathfrak{gl}_n$. Denote by x_{ij} the coordinates on $\mathfrak{g}$ that correspond to basis $\{E_{ij}\}_{i,j=1}^n$, i.e. $x_{ij} \in \mathfrak{g}^*$ such that for any $X \in \mathfrak{gl}_n$

$$X = \sum_{i,j=1}^{n} x_{ij}(X) \cdot E_{ij},$$

or equivalently

$$X = \begin{pmatrix} x_{11}(X) & x_{12}(X) & \dots & \dots & x_{1n}(X) \\ x_{21}(X) & x_{22}(X) & \dots & \dots & x_{2n}(X) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{n1}(X) & x_{n2}(X) & \dots & \dots & x_{nn}(X) \end{pmatrix}.$$

Clearly, for any $X, Y \in \mathfrak{gl}_n$ we have

$$B(X,Y) = \sum_{i,j=1}^{n} x_{ij}(X) x_{ji}(Y).$$

In particular, the basis of $\mathfrak{gl}_n$ dual to $\{E_{ij}\}_{i,j=1}^n$ with respect to B is $\{E_{ji}\}_{i,j=1}^n$.

2.4. **Notations and conventions.** Fix positive integers n and k such that $0 \le k \le n$. Let $\mathcal{I}_k$ be the set of all k-element subsets of the set $\{1, 2, \dots, n\}$. It is clear that $|\mathcal{I}_k| = \binom{n}{k}$. Let $\{e_1, e_2, \dots, e_n\}$ be a standard basis of $V = \mathbb{C}^n$. For each $I \in \mathcal{I}_k$, where $I = \{i_1, \dots, i_k\}$ and $i_1 < \dots < i_k$, denote by e_I the element $e_{i_1} \wedge \dots \wedge e_{i_k}$ of the k-th exterior power of V. Clearly, the elements $\{e_I\}_{I \in \mathcal{I}_k}$ form a basis of $\bigwedge^k(V)$.

Recall that if V is considered as a standard $\mathfrak{gl}_n$-module with the highest weight ω_1, then $V_{\omega_k} = \bigwedge^k(V)$ is a simple $\mathfrak{gl}_n$-module with the highest weight ω_k. The action of $\mathfrak{gl}_n$ is given via the formula

$$\pi_{\omega_k}(X)(v_1 \wedge v_2 \wedge \dots \wedge v_k) = (\pi_{\omega_1}(X)v_1) \wedge v_2 \wedge \dots \wedge v_k + \dots + v_1 \wedge v_2 \wedge \dots \wedge (\pi_{\omega_1}(X)v_k) \tag{2.9}$$

for any $X \in \mathfrak{gl}_n$.

In particular, for matrix units $\{E_{ij}\}_{i,j=1}^n$ of $\mathfrak{gl}_n$ and basis $\{e_l\}_{l=1}^n$ we have

$$\pi_{\omega_1}(E_{ij})e_l = \delta_{jl} e_i = \begin{cases} e_i, & \text{if } j = l, \\ 0, & \text{if } j \ne l. \end{cases}$$

Now define the coefficients $\varepsilon_{j,i}(J,I)$ for any $I, J \in \mathcal{I}_k$ and $i, j \in \{1, 2, \dots, n\}$ via the identity:

$$\pi_{\omega_k}(E_{ij})e_J = \sum_{I \in \mathcal{I}_k} \varepsilon_{j,i}(J,I) e_I.$$

One can check that the following statement holds.

Lemma 2.2. *Let $I, J \in \mathcal{I}_k$ and suppose that*

$$I = \{i_1, \dots, i_k\}, \ J = \{j_1, \dots, j_k\}, \ \text{where } i_1 < \dots < i_k, \ \text{and } j_1 < \dots < j_k.$$

Then, for any $i, j \in \{1, 2, \dots, n\}$ the coefficient $\varepsilon_{i,j}(I,J)$ can be found explicitly via the formula

$$\varepsilon_{j,i}(J,I) = \begin{cases} (-1)^m, & \text{if } i \in I, \ j \in J \text{ and } I \setminus \{i\} = J \setminus \{j\}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

Here in the first case m is equal to the number of inversions in the permutation

$$j_1, \dots, j_{r-1}, i, j_{r+1}, \dots, j_k$$

of the sequence $i_1, \dots, i_k$, where the index $r \in \{1, 2, \dots, k\}$ is such that $j_r = j$.

Recall that the elements of $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ can be regarded as polynomial $d(\omega_k) \times d(\omega_k)$-matrix-valued functions on $\mathfrak{g}$ (recall that we identify $\operatorname{End} V_{\omega_k}$ with the space of $d(\omega_k) \times d(\omega_k)$ matrices). Since elements of basis V_{ω_k} are enumerated by k-element subsets of $\{1, 2, \dots, n\}$ we can parameterize matrix elements of $A \in \mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ by pairs of elements of $\mathcal{I}_k$.

2.5. **Ordering on $\mathcal{I}_k$.** Define a natural ordering on $\mathcal{I}_k$ as follows: for any k-element subsets $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$, where $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$, we say that $I \preceq J$ if either

$$i_1 + \dots + i_k < j_1 + \dots + j_k$$

or there exists $s \le k$ such that

$$i_1 + \dots + i_r \le j_1 + \dots + j_r$$

for all $r = \overline{1,s}$ with a strict inequality for $r = s$. We use the introduced ordering on $\mathcal{I}_k$ to enumerate vectors of basis $\{e_I\}_{I \in \mathcal{I}_k}$. We identify each element $\operatorname{End} V_{\omega_k}$ with its $d(\omega_k) \times d(\omega_k)$ matrix in basis $\{e_I\}_{I \in \mathcal{I}_k}$.

3. Classical case

3.1. **Generators of Kirillov algebra.** Recall that we identified $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ with GL_n-equivariant polynomial maps on $\mathfrak{gl}_n$. Thus, we can regard elements of $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ as matrices whose entries are polynomials in x_{ij}. Define certain distinguished elements of $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ as follows: set

$$Y_m = \pi_{\omega_k}(X^m) = \sum_{i,j=1}^{n} \pi_{\omega_k}(E_{ij}) \cdot [X^m]_{ij}. \tag{3.1}$$

Here $X = \{x_{ij}\}_{i,j=1}^n$ is regarded as $\mathfrak{gl}_n$-valued polynomial function on $\mathfrak{gl}_n$. We also put $Y = Y_1$.

In particular, for any $I, J \in \mathcal{I}_k$ we have

$$[Y_m]_{IJ} = \sum_{i,j=1}^{n} \varepsilon_{i,j}(I,J) \cdot [X^m]_{ij}.$$

Lemma 3.1. *For any positive integer m the elements Y_m and $\mathrm{tr}(Y_m)$ belong to $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$.*

Proof. This can be proved by checking directly the condition (1.2). □

We will show later that elements Y_m and $\mathrm{tr}(Y_m)$ generate the whole algebra $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ (see Proposition 3.4).

3.2. **Equivariant cohomology of the complex Grassmannian.** Consider the complex Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional subspaces in a n-dimensional complex vector space. We will consider the GL_n-equivariant cohomology ring $H^*_{\mathrm{GL}_n}(\mathrm{Gr}(k,n),\mathbb{C})$ of $\mathrm{Gr}(k,n)$ with complex coefficients. It is known that the GL_n-equivariant cohomology of the complex Grassmannian of k-planes in n-dimensional vector space has the following algebraic presentation:

$$H^*_{\mathrm{GL}_n}(\mathrm{Gr}(k,n),\mathbb{C}) = \mathbb{C}[p_1,\ldots,p_k,q_1,\ldots,q_{n-k},a_1,\ldots,a_n]/\mathrm{coeffs}(p(t)q(t)-a(t)), \tag{3.2}$$

where $\mathrm{coeffs}(p(t)q(t)-a(t))$ is the ideal in $\mathbb{C}[p_1,\ldots,p_k,q_1,\ldots,q_{n-k}]$ generated by the coefficients of the polynomial $p(t)q(t)-a(t)$, where we define

$$p(t) = t^k + p_1 t^{k-1} + \ldots + p_k,\ \ q(t) = t^{n-k} + q_1 t^{n-k-1} + \ldots + q_{n-k},$$
$$a(t) = t^n + a_1 t^{n-1} + \ldots + a_1 t + a_0.$$

In other words, the ideal I is generated by the elements

$$a_r - \sum_{i+j=r} p_i q_j,\ \ r = \overline{1,n}.$$

In particular, the corresponding ring is commutative. It can be shown that as complex algebra it is isomorphic to the polynomial algebra $\mathbb{C}[p_1,\ldots,p_k,q_1,\ldots,q_{n-k}]$.

The following isomorphism is proved in [8]:

$$\mathcal{C}_{\omega_k}(\mathfrak{gl}_n) \simeq H^*_{\mathrm{GL}_n}(\mathrm{Gr}(k,n),\mathbb{C}) \simeq \mathbb{C}[p_1,\ldots,p_k,q_1,\ldots,q_{n-k},a_1,\ldots,a_n]/\mathrm{coeffs}(p(t)q(t)-a(t)). \tag{3.3}$$

Remark 3.1. This is one instance of the observation due to Panyushev that certain classical family algebras are isomorphic to equivariant cohomology rings of algebraic varieties. In particular, in the case of minuscule representations the corresponding variety is a generaized flag variety. We refer the reader to [11, Section 6] for more information.

On the other hand, we also have the following isomorphism of algebras

$$\mathbb{C}[p_1,\ldots,p_k,q_1,\ldots,q_{n-k},a_1,\ldots,a_n]/\mathrm{coeffs}(p(t)q(t)-a(t)) \simeq \mathbb{C}[t_1,\ldots,t_k,t_{k+1},\ldots,t_n]^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}, \tag{3.4}$$

where the group $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ acts naturally on $t_1,\ldots,t_k,t_{k+1},\ldots,t_n$ i.e. by permuting separately the first k variables and the last $n-k$ variables. The last isomorphism acts on the generators p_i, q_j, a_r as follows:

$$p_i \mapsto (-1)^i e_i(t_1\ldots,t_k),\ \ q_j \mapsto (-1)^j e_j(t_{k+1},\ldots,t_n),\ \ a_r \mapsto (-1)^r e_r(t_1,\ldots,t_k,t_{k+1},\ldots,t_n). \tag{3.5}$$

Here e_l is the l-th elementary symmetric polynomial (see [3] for more information about the symmetric functions).

We discuss the isomorphism between $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ and $\mathbb{C}[t_1,\ldots,t_k,t_{k+1},\ldots,t_n]^{\mathfrak{S}_k\times\mathfrak{S}_{n-k}}$ in more detail in Subsection 3.4. We obtain this isomorphism independently using only algebraic methods.

3.3. **Diagonalization homomorphism.** Consider the following evaluation map from $\mathbb{C}[x_{ij},\, i,j=\overline{1,n}]$ to $\mathbb{C}[t_1,\dots,t_n]$: for any polynomial F in x_{ij} define

$$\Psi(F) = F|_{x_{ij}=\delta_{ij}t_i}.$$

Recall that we regard the elements $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ as $d(\omega_k)\times d(\omega_k)$-matrices with entries in $\mathbb{C}[x_{ij},\, i,j=\overline{1,n}]$. By extending the map Ψ entry-wise onto $\mathrm{Mat}(d(\omega_k),\mathbb{C}[x_{ij},\, i,j=\overline{1,n}])$ we can define the homomorphism Diag on $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ as follows:

$$\mathrm{Diag}\colon \mathcal{C}_{\omega_k}(\mathfrak{gl}_n)\to \mathrm{Mat}(d(\omega_k),\mathbb{C}[t_1,\dots,t_n]),$$

where

$$\mathrm{Diag}(A)=\Psi(A).$$

In other words, we evaluate the matrix-valued polynomial function A in variables x_{ij} at $x_{ii}=t_i$, $x_{ij}=0$ for $i\neq j$.

Remark 3.2. One can regard $t_1,\dots,t_n$ as coordinates on the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{gl}_n$, i.e. the abelian subalgebra consisting of the diagonal matrices. Then, one can view $\mathrm{Diag}(A)$ as a restriction of A to $\mathfrak{h}$.

3.4. **The maps ψ_I.** For any $I\in\mathcal{I}_k$ denote by ψ_I the map

$$\psi_I\colon \mathcal{C}_{\omega_k}(\mathfrak{gl}_n)\to\mathbb{C}[t_1,\dots,t_n],\ \ \psi_I(A)=[\mathrm{Diag}(A)]_{II}.$$

Proposition 3.2. *(a) For any $I\in\mathcal{I}_k$, $\sigma\in\mathfrak{S}_n$ and any $A\in\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ we have*

$$\psi_{\sigma(I)}(A)=\sigma(\psi_I(A)). \tag{3.6}$$

(b) The image of Diag *consists of diagonal matrices and for any $I\in\mathcal{I}_k$ the map ψ_I is an algebra isomorphism between $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ and $\mathbb{C}[t_1,\dots,t_n]^{\mathrm{Stab}_n(I)}$.*

Remark 3.3. The symmetric group $\mathfrak{S}_n$ naturally acts on $\{1,2,\dots,n\}$. This gives rise to a natural $\mathfrak{S}_n$-action on the family $\mathcal{I}_k$ of all k-element subsets of $\{1,2\dots,n\}$. Note that for any $I\in\mathcal{I}_k$ the stabilizer of I under the action of $\mathfrak{S}_n$ is isomorphic to $\mathfrak{S}_k\times\mathfrak{S}_{n-k}$. We denote this subgroup of $\mathfrak{S}_n$ as $\mathrm{Stab}_n(I)$.

Remark 3.4. The symmetric group $\mathfrak{S}_n$ naturally acts on $\mathbb{C}[t_1,\dots,t_n]$ via

$$(\sigma f)(t_1,\dots,t_n)=f(t_{\sigma(1)},\dots,t_{\sigma(n)})$$

and thus we can consider the $\mathrm{Stab}_n(I)$-invariant part of $\mathbb{C}[t_1,\dots,t_n]$.

Proof. Recall that elements of $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ can be characterized by the following condition

$$\pi_{\omega_k}(g)A(X)\pi_{\omega_k}(g)^{-1}=A(\mathrm{Ad}(g)(X)) \text{ for all } g\in\mathrm{GL}_n,\ X\in\mathfrak{gl}_n. \tag{3.7}$$

It follows that for $X\in\mathfrak{h}$ and diagonal invertible matrices g we have

$$\pi_{\omega_k}(g)A(X)\pi_{\omega_k}(g)^{-1}=A(X), \tag{3.8}$$

and hence

$$\pi_{\omega_k}(g)\,\mathrm{Diag}(A)\pi_{\omega_k}(g)^{-1}=\mathrm{Diag}(A).$$

Therefore, for any $Y\in\mathfrak{h}$ we have

$$[\pi_{\omega_k}(Y),\mathrm{Diag}(A)]=0.$$

Since all weights of the representation π_{ω_k} are distinct, the latter implies that $\mathrm{Diag}(A)$ is a diagonal $d(\omega_k)\times d(\omega_k)$-matrix.

To prove the second part of the proposition we take $X\in\mathfrak{h}$ and $g=\rho(\sigma)$ in (3.7), where $\sigma\in\mathfrak{S}_n$ and $\rho\colon\mathfrak{S}_n\to\mathrm{GL}_n$ is a standard representation of $\mathfrak{S}_n$. It follows that for any $I\in\mathcal{I}_k$ and any $\sigma\in\mathfrak{S}_n$ we have

$$[\mathrm{Diag}(A)]_{\sigma(I)\sigma(I)}=\sigma\big([\mathrm{Diag}(A)]_{II}\big),\ \text{ i.e. } \psi_{\sigma(I)}(A)=\sigma(\psi_I(A)). \tag{3.9}$$

Thus, $\mathrm{Diag}(A)\in\mathbb{C}[t_1,\dots,t_n]^{\mathrm{Stab}_n(I)}$.

Since $\mathrm{Diag}(A)$ is a diagonal matrix and Ψ is homomorphism, the map $\psi_I\colon A\mapsto[\mathrm{Diag}(A)]_{II}$ is an algebra homomorphism from $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ to $\mathbb{C}[t_1,\dots,t_n]^{\mathrm{Stab}_n(I)}$.

Now let us prove that the ψ_I is injective for any $I\in\mathcal{I}_k$. Indeed, assume that for some $A\in\mathcal{C}_{\omega_k}(\mathfrak{g})$ we have $[\mathrm{Diag}(A)]_{II}=0$. It follows that $\Psi(A)=0$. In other words, the polynomial function $A\colon\mathfrak{gl}_n\to\mathrm{Mat}(d(\omega)\times d(\omega_k))$ vanishes on $\mathfrak{h}$. The condition (3.8) implies that A also vanishes on the Zariski dense subset of semisimple elements of $\mathfrak{gl}_n$. Thus, $A=0$.

Finally, to prove the surjectivity of the map ψ_I we need the following lemma.

Lemma 3.3. *(a) The map* Diag *acts on elements Y_m and $\mathrm{tr}(Y^m)$ as follows:*

$$\psi_I(Y_\beta)=\sum_{i\in I}t_i^\beta,\ \ \psi_I(\mathrm{tr}(Y^\alpha))=\sum_{J\in\mathcal{I}_k}\Big(\sum_{i\in J}t_i\Big)^\alpha. \tag{3.10}$$

(b) The elements $\sum_{i\in I}t_i^\beta$, $\beta=\overline{0,k}$ and $\sum_{J\in\mathcal{I}_k}\big(\sum_{i\in J}t_i\big)^r$, $\alpha=\overline{1,n-k}$ generate the algebra $\mathbb{C}[t_1,\dots,t_n]^{\mathrm{Stab}_n(I)}$.

Proof. The part (a) of the lemma follows from direct computations and the part (b) is a consequence of some general facts about symmetric functions. □

Remark 3.5. In order to obtain the isomorphism mentioned in Subsection 3.2 we just need to take $I = \{1, 2, \dots, k\}$.

Since elements Y_m and $\mathrm{tr}(Y^m)$ belong to $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ and their images generate $\mathbb{C}[t_1, \dots, t_n]^{\mathrm{Stab}_n(I)}$, the map $A \mapsto [\mathrm{Diag}(A)]_{II}$ is surjective. □

In the course of the proof of Proposition 3.2 we also proved the following fact.

Proposition 3.4. *Kirillov algebra $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ is generated by elements Y_m, $m \geq 0$ and traces of powers of Y. Moreover,*

$$\mathcal{C}_{\omega_k}(\mathfrak{gl}_n) = \left\langle Y_0, Y_1, \dots, Y_k, \mathrm{tr}(Y), \dots, \mathrm{tr}(Y^{n-k}) \right\rangle. \tag{3.11}$$

3.5. **Derivation of the formula for D.** In case of $\mathfrak{g} = \mathfrak{gl}_n$ we define an invariant form as follows:

$$B(X, Y) = \mathrm{tr}(XY) = \sum_{i,j=1}^{n} x_{ij}(X) x_{ji}(Y). \tag{3.12}$$

In particular, the dual basis of $\{E_{ij}\}_{i,j=1}^n$ with the respect to B is $\{E_{ji}\}_{i,j=1}^n$. Therefore, the action of D-operator can now be rewritten as follows:

$$D(A) = \sum_{i,j=1}^{n} \pi_{\omega_k}(E_{ji}) \cdot \frac{\partial A}{\partial E_{ij}}. \tag{3.13}$$

From now on we interpret matrix elements of A as polynomials in variables x_{ij} (which in turn are elements of $\mathfrak{gl}_n^*$). Then, it is clear that $\frac{\partial A}{\partial E_{ij}}$ corresponds to $\frac{\partial A}{\partial x_{ij}}$. For any $J, K \in \mathcal{I}_k$ the matrix element $[D(A)]_{JK}$ of $D(A)$ can be expressed as follows:

$$[D(A)]_{JK} = \sum_{i,j=1}^{n} \sum_{I \in \mathcal{I}_k} [\pi_{\omega_k}(E_{ji})]_{JI} \cdot \left[\frac{\partial A}{\partial x_{ij}}\right]_{IK} = \sum_{i,j=1}^{n} \sum_{I \in \mathcal{I}_k} \varepsilon_{ij}(I, J) \cdot \frac{\partial A_{IK}}{\partial x_{ij}}. \tag{3.14}$$

We use the isomorphism from Proposition 3.2.

Proposition 3.5. *Let $I \in \mathcal{I}_k$ and let $I' = \{1, 2, \dots, n\} \setminus I$. Then, the operator*

$$\psi_I \circ D \circ \psi_I^{-1} \colon \mathbb{C}[t_1, \dots, t_n]^{\mathrm{Stab}_n(I)} \to \mathbb{C}[t_1, \dots, t_n]^{\mathrm{Stab}_n(I)}$$

acts as the following operator

$$\sum_{i \in I} \frac{\partial}{\partial t_i} + \sum_{i \in I,\, j \in I'} \partial_{ij}.$$

Remark 3.6. Note this operator is not differential, i.e. does not satisfy the Leibniz rule.

To prove Proposition 3.5 we need two technical lemmas.

Lemma 3.6. *For any indices i, j and any positive integer β we have*

$$\frac{\partial}{\partial x_{ij}} \left[X^\beta\right] = \sum_{\gamma=0}^{\beta-1} X^\gamma E_{ij} X^{\beta-\gamma-1}. \tag{3.15}$$

In particular, for any indices i', j' we have

$$\Psi\left(\frac{\partial}{\partial x_{ij}} \left[X^\beta\right]_{i'j'}\right) = \begin{cases} \delta_{ii'}\delta_{jj'} \cdot \frac{t_i^\beta - t_j^\beta}{t_i - t_j}, & i \neq j, \\ \delta_{ii'}\delta_{jj'} \cdot \beta t_i^{\beta-1}, & i = j. \end{cases} \tag{3.16}$$

Proof. We have

$$\frac{\partial}{\partial x_{ij}} \left[X^\beta\right] = \sum_{\gamma=0}^{\beta-1} X^\gamma \cdot \left(\frac{\partial}{\partial x_{ij}} X\right) \cdot X^{\beta-\gamma-1} = \sum_{\gamma=0}^{\beta-1} X^\gamma E_{ij} X^{\beta-\gamma-1} \tag{3.17}$$

and hence,

$$\Psi\left(\frac{\partial}{\partial x_{ij}} [X^\beta]_{i'j'}\right) = \sum_{\gamma=0}^{\beta-1} \left[\Psi(X)^\gamma\, \Psi(E_{ij}) \Psi(X)^{\beta-\gamma-1}\right]_{i'j'} = \delta_{ii'}\delta_{jj'} \cdot \sum_{\gamma=0}^{\beta-1} t_i^\gamma t_j^{\beta-\gamma-1}$$

which is equivalent to the formula above. □

Lemma 3.7. *For any indices i, j, any $I, J \in \mathcal{I}_k$ and any positive integers α, β we have*

$$\Psi\left(\frac{\partial}{\partial x_{ij}}[Y_\beta]_{IJ}\right) = \begin{cases} \varepsilon_{i,j}(I,J)\cdot \frac{t_i^\beta - t_j^\beta}{t_i - t_j}, & i \neq j, \\ \varepsilon_{i,j}(I,J)\cdot \beta t_i^{\beta-1}, & i = j. \end{cases} \tag{3.18}$$

and

$$\Psi\left(\frac{\partial}{\partial x_{ij}}[Y_\beta^\alpha]_{IJ}\right) = \begin{cases} \varepsilon_{i,j}(I,J)\cdot \frac{1}{t_i - t_j}\left((\sum_{p\in I} t_p^\beta)^\alpha - (\sum_{p\in J} t_p^\beta)^\alpha\right), \ i \neq j \\ \varepsilon_{i,j}(I,J)\cdot \beta t_i^{\beta-1}(\sum_{p\in I} t_p^\beta)^{\alpha-1}, \ i = j. \end{cases} \tag{3.19}$$

In other words,

$$\Psi\left(\frac{\partial}{\partial x_{ij}}[Y_\beta^\alpha]_{IJ}\right) = \begin{cases} \varepsilon_{i,j}(I,J)\cdot \partial_{ij}(\psi_I(Y_\beta^\alpha)), \ i \neq j \\ \varepsilon_{i,j}(I,J)\cdot \frac{\partial}{\partial t_i}(\psi_I(Y_\beta^\alpha)), \ i = j, \end{cases} \tag{3.20}$$

Proof. Note that

$$\Psi\left(\frac{\partial}{\partial x_{ij}}[Y_\beta]_{IJ}\right) = \sum_{i',j'=1}^n \varepsilon_{i',j'}(I,J)\cdot \Psi\left(\frac{\partial}{\partial x_{ij}}[X^\beta]_{i'j'}\right)$$

Applying Lemma 3.6 we obtain the first equality. For the second one note that

$$\Psi\left(\frac{\partial}{\partial x_{ij}}[Y_\beta^\alpha]_{IJ}\right) = \sum_{\gamma=0}^{\alpha-1}[\Psi(Y_\beta)^\gamma]_{II}\left[\Psi\left(\frac{\partial}{\partial x_{ij}}Y_\beta\right)\right]_{IJ}[\Psi(Y_\beta)^{\alpha-\gamma-1}]_{JJ} =$$
$$= \sum_{\gamma=0}^{\alpha-1}\left(\sum_{p\in I} t_p^\beta\right)^\gamma \left[\Psi\left(\frac{\partial}{\partial x_{ij}}Y_\beta\right)\right]_{IJ}\left(\sum_{p\in J} t_p^\beta\right)^{\alpha-\gamma-1}.$$

If $i \neq j$, then we can rewrite the last expression as

$$\sum_{\gamma=0}^{\alpha-1}\left(\sum_{p\in I} t_p^\beta\right)^\gamma \left[\Psi\left(\frac{\partial}{\partial x_{ij}}Y_\beta\right)\right]_{IJ}\left(\sum_{p\in J} t_p^\beta\right)^{\alpha-\gamma-1} = \varepsilon_{i,j}(I,J)\,\frac{t_i^\beta - t_j^\beta}{t_i - t_j}\cdot\sum_{\gamma=0}^{\alpha-1}\left(\sum_{p\in I} t_p^\beta\right)^\gamma\left(\sum_{p\in J} t_p^\beta\right)^{\alpha-\gamma-1} =$$
$$= \varepsilon_{i,j}(I,J)\,\frac{t_i^\beta - t_j^\beta}{t_i - t_j}\cdot\frac{\left(\sum_{p\in I} t_p^\beta\right)^\alpha - \left(\sum_{p\in I} t_p^\beta\right)^\alpha}{\sum_{p\in J} t_p^\beta - \sum_{p\in J} t_p^\beta}.$$

Now note that if $\varepsilon_{i,j}(I,J) \neq 0$, then we have $i \in I$, $j \in J$ and $I \setminus \{i\} = J \setminus \{j\}$ (see Lemma 2.2). Therefore, if $i \neq j$, then

$$\Psi\left(\frac{\partial}{\partial x_{ij}}[Y_\beta^\alpha]_{IJ}\right) = \varepsilon_{i,j}(I,J)\,\frac{\left(\sum_{p\in I} t_p^\beta\right)^\alpha - \left(\sum_{p\in J} t_p^\beta\right)^\alpha}{t_i - t_j} = \varepsilon_{i,j}(I,J)\,\partial_{ij}\left(\left(\sum_{p\in I} t_p^\beta\right)^\alpha\right).$$

If $i = j$, then $\varepsilon_{i,j}(I,J) \neq 0$ only if $I = J$ and $i \in I$ and in this case

$$\sum_{\gamma=0}^{\alpha-1}\left(\sum_{p\in I} t_p^\beta\right)^\gamma \left[\Psi\left(\frac{\partial}{\partial x_{ij}}Y_\beta\right)\right]_{IJ}\left(\sum_{p\in J} t_p^\beta\right)^{\alpha-\gamma-1} = \varepsilon_{i,j}(I,J)\cdot\beta t_i^{\beta-1}\cdot\alpha\left(\sum_{p\in I} t_p^\beta\right)^{\alpha-1} =$$
$$= \varepsilon_{i,j}(I,J)\,\frac{\partial}{\partial t_i}\left(\left(\sum_{p\in I} t_p^\beta\right)^\alpha\right).$$

To conclude the proof it remains to note that $\psi_I(Y_\beta^\alpha) = \left(\sum_{p\in I} t_p^\beta\right)^\alpha$ (see Lemma 3.3). □

Lemma 3.8. *Let i and j be distinct elements of $\{1,\dots,n\}$. Then, for any polynomials $\{f_r\}_{r=1}^m$ in $\mathbb{C}[t_1,\dots,t_n]$ the following identity holds*

$$\partial_{ij}\left(\prod_{r=1}^m f_r\right) = \sum_{r=1}^m s_{ij}\left(\prod_{q<r} f_q\right)\partial_{ij}(f_r)\left(\prod_{q>r} f_q\right). \tag{3.21}$$

Proof. For $m = 2$ it follows from

$$\partial_{ij}(fg) = \frac{fg - s_{ij}(fg)}{t_i - t_j} = \frac{(f - s_{ij}(f))g + s_{ij}(f)(g - s_{ij}(g))}{t_i - t_j} = \partial_{ij}(f)g + s_{ij}(f)\partial_{ij}(g).$$

Induction on m now gives the general case. □

Remark 3.7. This identity may be considered as an analogue of Leibniz rule for divided differences.

Proof. In order to recover the action of D it suffices to understand how it acts on the basis elements. Suppose that

$$A = \prod_{r=1}^{l} \operatorname{tr}(Y^{\alpha_r}) \cdot \prod_{s=1}^{m} Y_{\beta_s}. \tag{3.22}$$

Then

$$[D(A)]_{JK} = \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J) \cdot \frac{\partial A_{IK}}{\partial x_{ij}} = \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J) \cdot \frac{\partial}{\partial x_{ij}} \left(\prod_{r=1}^{l} \operatorname{tr}(Y^{\alpha_r}) \cdot \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right) = \tag{3.23}$$

$$= \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J) \left(\prod_{r=1}^{l} \operatorname{tr}(Y^{\alpha_r}) \cdot \frac{\partial}{\partial x_{ij}} \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} + \frac{\partial}{\partial x_{ij}} \left(\prod_{r=1}^{l} \operatorname{tr}(Y^{\alpha_r}) \right) \cdot \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right). \tag{3.24}$$

Thus,

$$\begin{aligned}
\Psi([D(A)]_{JK}) &= \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J) \prod_{r=1}^{l} \Psi(\operatorname{tr}(Y^{\alpha_r})) \cdot \Psi\left(\frac{\partial}{\partial x_{ij}} \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right) + \\
&+ \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J)\, \Psi\left(\frac{\partial}{\partial x_{ij}} \left(\prod_{r=1}^{l} \operatorname{tr}(Y^{\alpha_r}) \right) \right) \cdot \left[\Psi\left(\prod_{s=1}^{m} Y_{\beta_s} \right) \right]_{IK} = \\
&= \prod_{r=1}^{l} \Psi(\operatorname{tr}(Y^{\alpha_r})) \cdot \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J)\, \Psi\left(\frac{\partial}{\partial x_{ij}} \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right) + \\
&+ \left[\Psi\left(\prod_{s=1}^{m} Y_{\beta_s} \right) \right]_{KK} \cdot \sum_{i,j=1}^{n} \varepsilon_{ij}(K,J)\, \Psi\left(\frac{\partial}{\partial x_{ij}} \left(\prod_{r=1}^{l} \operatorname{tr}(Y^{\alpha_r}) \right) \right),
\end{aligned}$$

or

$$\Psi([D(A)]_{JK}) = \prod_{r=1}^{l} \Psi(\operatorname{tr}(Y^{\alpha_r})) \cdot S_\beta(J,K) + \left(\prod_{s=1}^{m} \psi_K(Y_{\beta_s}) \right) \cdot S_\alpha(J,K),$$

where

$$S_\alpha(J,K) = \sum_{i,j=1}^{n} \varepsilon_{ij}(K,J)\, \Psi\left(\frac{\partial}{\partial x_{ij}} \left(\prod_{r=1}^{l} \operatorname{tr}(Y^{\alpha_r}) \right) \right), \tag{3.25}$$

$$S_\beta(J,K) = \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J)\, \Psi\left(\frac{\partial}{\partial x_{ij}} \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right). \tag{3.26}$$

Here we used the fact that Ψ maps $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ to diagonal matrices.

We now simplify these two expressions $S_\alpha(J,K)$ and $S_\beta(J,K)$ separately. Note that

$$\Psi\left(\frac{\partial}{\partial x_{ij}} \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right) = \left[\sum_{s=1}^{m} \left(\prod_{1\le r<s} \Psi(Y_{\beta_r}) \right) \Psi\left(\frac{\partial}{\partial x_{ij}} Y_{\beta_s} \right) \left(\prod_{s<r\le m} \Psi(Y_{\beta_r}) \right) \right]_{IK}.$$

However, all matrices $\Psi(Y_\beta)$ are diagonal and hence,

$$\begin{aligned}
\Psi\left(\frac{\partial}{\partial x_{ij}} \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right) &= \sum_{s=1}^{m} \left(\prod_{1\le r<s} [\Psi(Y_{\beta_r})]_{II} \right) \left[\Psi\left(\frac{\partial}{\partial x_{ij}} Y_{\beta_s} \right) \right]_{IK} \left(\prod_{s<r\le m} [\Psi(Y_{\beta_r})]_{KK} \right) = \\
&= \sum_{s=1}^{m} \left(\prod_{r<s} \psi_I(Y_{\beta_r}) \right) \cdot \Psi\left(\frac{\partial}{\partial x_{ij}} [Y_{\beta_s}]_{IK} \right) \cdot \left(\prod_{r>s} \psi_K(Y_{\beta_r}) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_\beta(J,K) &= \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J)\, \Psi\left(\frac{\partial}{\partial x_{ij}} \left[\prod_{s=1}^{m} Y_{\beta_s} \right]_{IK} \right) = \\
&= \sum_{i,j=1}^{n} \sum_{I\in\mathcal{I}_k} \varepsilon_{ij}(I,J) \sum_{s=1}^{m} \left(\prod_{r<s} \psi_I(Y_{\beta_r}) \right) \cdot \Psi\left(\frac{\partial}{\partial x_{ij}} [Y_{\beta_s}]_{IK} \right) \cdot \left(\prod_{r>s} \psi_K(Y_{\beta_r}) \right)
\end{aligned}$$

Lemma 3.7 implies that $\Psi\left(\frac{\partial}{\partial x_{ij}} [Y_{\beta_s}]_{IK} \right)$ is zero unless $\varepsilon_{i,j}(I,K) = 0$. Note that if $J \neq K$, then $\varepsilon_{i,j}(I,J)\varepsilon_{i,j}(I,K) = 0$ for all $i,j \in \{1,\dots,n\}$ and $I \in \mathcal{I}_k$. Thus, $S_\beta(J,K) = 0$ if $J \neq K$.

Now assume that $J = K$. Then, we can split the first summation in $S_\beta(J,K)$ and apply Lemma 3.7 as follows:

$$S_\beta(J,J) = \sum_{i=1}^{n}\sum_{I\in\mathcal{I}_k}\varepsilon_{ii}(I,J)^2\sum_{s=1}^{m}\left(\prod_{r<s}\psi_I(Y_{\beta_r})\right)\frac{\partial}{\partial t_i}(\psi_I(Y_{\beta_s}))\left(\prod_{r>s}\psi_J(Y_{\beta_r})\right)+$$
$$+\sum_{i\neq j}\sum_{I\in\mathcal{I}_k}\varepsilon_{ij}(I,J)^2\sum_{s=1}^{m}\left(\prod_{r<s}\psi_I(Y_{\beta_r})\right)\partial_{ij}(\psi_I(Y_{\beta_s}))\left(\prod_{r>s}\psi_J(Y_{\beta_r})\right).$$

Since $\varepsilon_{ij}(I,J) = 0$ unless $i\in I$, $j\in J$ and $I\setminus\{i\} = J\setminus\{j\}$, we have (here we denote $J' = \{1,\ldots,n\}\setminus J$)

$$S_\beta(J,J) = \left(\sum_{i\in J}\frac{\partial}{\partial t_i}\right)\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right)+\sum_{\substack{i\in J'\\ j\in J}}\sum_{s=1}^{m}\left(\prod_{r<s}\psi_{s_{ij}(J)}(Y_{\beta_r})\right)\partial_{ij}(\psi_{s_{ij}(J)}(Y_{\beta_s}))\left(\prod_{r>s}\psi_J(Y_{\beta_r})\right)=$$
$$=\left(\sum_{i\in J}\frac{\partial}{\partial t_i}\right)\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right)+\sum_{\substack{i\in J'\\ j\in J}}\sum_{s=1}^{m}s_{ij}\left(\prod_{r<s}\psi_J(Y_{\beta_r})\right)\partial_{ij}(\psi_{s_{ij}(J)}(Y_{\beta_s}))\left(\prod_{r>s}\psi_J(Y_{\beta_r})\right)=$$
$$=\left(\sum_{i\in J}\frac{\partial}{\partial t_i}\right)\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right)+\sum_{\substack{i\in J'\\ j\in J}}\partial_{ji}\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right).$$

It remains to simplify the sum $S_\alpha(J,K)$. Firstly, Lemma 3.7 implies that

$$\Psi\left(\frac{\partial}{\partial x_{ij}}\operatorname{tr}(Y^{\alpha_r})\right) = \sum_{L\in\mathcal{I}_k}\Psi\left(\frac{\partial}{\partial x_{ij}}[Y^{\alpha_r}]_{LL}\right).$$

Hence, $\Psi\left(\frac{\partial}{\partial x_{ij}}\operatorname{tr}(Y^{\alpha_r})\right) = 0$ unless $\varepsilon_{i,j}(L,L)\neq 0$, i.e. $i=j$ and $i\in L$. Therefore,

$$S_\alpha(J,K) = \sum_{i,j=1}^{n}\varepsilon_{ij}(K,J)\,\Psi\left(\frac{\partial}{\partial x_{ij}}\left(\prod_{r=1}^{l}\operatorname{tr}(Y^{\alpha_r})\right)\right)=$$
$$=\sum_{i,j=1}^{n}\varepsilon_{ij}(K,J)\left(\sum_{r=1}^{l}\left(\prod_{q\neq r}\Psi(\operatorname{tr}(Y^{\alpha_q}))\right)\cdot\Psi\left(\frac{\partial}{\partial x_{ij}}\operatorname{tr}(Y^{\alpha_r})\right)\right)=$$
$$=\sum_{i=1}^{n}\varepsilon_{ii}(K,J)\left(\sum_{r=1}^{l}\left(\prod_{q\neq r}\Psi(\operatorname{tr}(Y^{\alpha_q}))\right)\left(\sum_{\substack{L\in\mathcal{I}_k\\ L\ni i}}\frac{\partial}{\partial t_i}(\psi_L(Y^{\alpha_r}))\right)\right).$$

Now note that for any $L\in\mathcal{I}_k$ such that $i\notin L$ we have $\psi_L(Y^\alpha) = \left(\sum_{l\in L}t_l\right)^\alpha$ and hence $\frac{\partial}{\partial t_i}(\psi_L(Y^\alpha)) = 0$. Besides that, if $J\neq K$, then $\varepsilon_{ii}(K,J) = 0$ for all i and $S_\alpha(J,K) = 0$. If $J = K$, then $\varepsilon_i i(J,J)$ equals 1 if $i\in J$ and 0 otherwise. Thus, in the case $J = K$ we can rewrite the expression above as follows:

$$S_\alpha(J,J) = \sum_{i\in J}\sum_{r=1}^{l}\left(\prod_{q\neq r}\Psi(\operatorname{tr}(Y^{\alpha_q}))\right)\left(\sum_{L\in\mathcal{I}_k}\frac{\partial}{\partial t_i}(\psi_L(Y^{\alpha_r}))\right)=$$
$$=\left(\sum_{i\in J}\frac{\partial}{\partial t_i}\right)\left(\prod_{r=1}^{l}\Psi(\operatorname{tr}(Y^{\alpha_r}))\right).$$

The formulas for $S_\alpha(J,K)$ and $S_\beta(J,K)$ show that $[D(A)]_{J,K}$ is zero if $J\neq K$. If $J = K$, then we have

$$\Psi([D(A)]_{JJ}) = \prod_{r=1}^{l}\Psi(\operatorname{tr}(Y^{\alpha_r}))\cdot S_\beta(J,J)+\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right)\cdot S_\alpha(J,J)= \tag{3.27}$$
$$=\prod_{r=1}^{l}\Psi(\operatorname{tr}(Y^{\alpha_r}))\cdot\left(\left(\sum_{i\in J}\frac{\partial}{\partial t_i}\right)\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right)+\sum_{\substack{i\in J'\\ j\in J}}\partial_{ji}\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right)\right)+ \tag{3.28}$$
$$+\left(\prod_{s=1}^{m}\psi_J(Y_{\beta_s})\right)\cdot\left(\sum_{i\in J}\frac{\partial}{\partial t_i}\right)\left(\prod_{r=1}^{l}\Psi(\operatorname{tr}(Y^{\alpha_r}))\right). \tag{3.29}$$

Since $\prod_{r=1}^{l} \Psi(\mathrm{tr}(Y^{\alpha_r}))$ is a symmetric polynomial in variables $t_1, \ldots, t_n$, we have

$$\partial_{ji}\left(\prod_{r=1}^{l} \Psi(\mathrm{tr}(Y^{\alpha_r}))\right) = 0$$

for all i, j. Finally, recall that $A = \prod_{r=1}^{l} \mathrm{tr}(Y^{\alpha_r}) \cdot \prod_{s=1}^{m} Y_{\beta_s}$. It follows that the formula for $[D(A)]_{JJ}$ above can be rewritten as

$$\psi_J(D(A)) = \Psi([D(A)]_{JJ}) = \left(\sum_{i\in J} \frac{\partial}{\partial t_i}\right)(\psi_J(A)) + \sum_{\substack{i\in J' \\ j\in J}} \partial_{ij}(\psi_J(A)).$$

Therefore, the linear operator $\psi_J \circ D \circ \psi_J^{-1}$ acts as $\sum_{i\in J} \frac{\partial}{\partial t_i} + \sum_{i\in J',\, j\in J} \partial_{ji}$ on basis elements of $\mathbb{C}[t_1, \ldots, t_n]^{\mathrm{Stab}_n(J)}$ and this concludes the proof. □

Remark 3.8. In the course of the proof we showed again that D maps $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ to itself.

3.6. **Relations in subalgebras generated by** D**.** Recall that the GL_n-equivariant cohomology ring of $\mathrm{Gr}(k, n)$ has the following presentation in terms of generators and relations:

$$H^*_{\mathrm{GL}_n}(\mathrm{Gr}(k, n), \mathbb{C}) = \mathbb{C}[p_1, \ldots, p_k, q_1, \ldots, q_{n-k}, a_1, \ldots, a_n]/\mathrm{coeffs}(p(t)q(t) - a(t)). \quad (3.30)$$

Using the isomorphism from Subsection 3.2 we can identify $H^*_{\mathrm{GL}_n}(\mathrm{Gr}(k, n), \mathbb{C})$ with Kirillov algebra $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$. It is known that under this identification the elements $\{a_i\}_{i=1}^{n}$ are generators of the ring $\mathbb{C}[\mathfrak{gl}_n]^{\mathfrak{gl}_n}$.

Remark 3.9. In fact, a_m as an invariant polynomial on $\mathfrak{gl}_n$ is just the m-th elementary symmetric polynomial in eigenvalues of the corresponding element in $\mathfrak{gl}_n$. This can also be seen from (3.5) and properties of the diagonalization map.

In particular, we can define the action of D-operator on $H^*_{\mathrm{GL}}(\mathrm{Gr}(k, n), \mathbb{C})$. It is known that for any classical family algebra the image $D(\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}})$ lies in the center of $\mathcal{C}_\mu(\mathfrak{g})$ (see [4, Theorem M]). In particular, it means that $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ and $D(\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}})$ generate a commutative subalgebra inside $\mathcal{C}_\mu(\mathfrak{g})$. We now can apply the results obtained for $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ to describe the corresponding commutative subalgebra in that case.

Using the isomorphism (3.4), formulas (3.5) and formula for the action of D one can obtain the following explicit formula:

$$D(a_r) = \sum_{i+j=r} (k-i)p_i q_j. \quad (3.31)$$

Using ideas from the elimination theory we can obtain explicit relations between $D(a_i)$. Indeed, recall that

$$a_r = \sum_{i+j=r} p_i q_j. \quad (3.32)$$

The relations (3.32) and (3.31) can be simplified if we add auxiliary variable t and extend the action of D on polynomials in t with coefficients in $H^*_{\mathrm{GL}}(\mathrm{Gr}(k, n), \mathbb{C})$:

$$D(a(t)) = p'(t)q(t),\ a(t) = p(t)q(t),$$

where

$$\begin{aligned} a(t) &= t^n + a_1 t^{n-1} + \ldots + a_n, \\ p(t) &= t^k + p_1 t^{k-1} + \ldots + p_k, \\ q(t) &= t^{n-k} + q_1 t^{n-k-1} + \ldots + q_{n-k}. \end{aligned}$$

For convenience we also define $a_0 = 1$, $p_0 = 1$ and $q_0 = 1$.

It follows that $a'(t) - D(a(t)) = p(t)q'(t)$ and $a(t) = p(t)q(t)$ have a common divisor $p(t)$ of the degree k. Then, the properties of the Sylvester matrix give the relations between elements a_r and $D(a_m)$.

Proposition 3.9. *Denote* $b_m = D(a_{m+1})$ *for all* $m = \overline{0, n-1}$*. Then, all* $(2n-k) \times (2n-k)$ *minors of the following* $(2n-1) \times (2n-1)$ *matrix*

$$\begin{pmatrix} a_0 & a_1 & \ldots & \ldots & a_{n-1} & a_n & 0 & \ldots & \ldots & 0 \\ 0 & a_0 & \ldots & \ldots & \ldots & a_{n-1} & a_n & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & a_0 & a_1 & a_2 & \ldots & \ldots & \ldots & a_n \\ na_0 - b_0 & (n-1)a_1 - b_1 & \ldots & \ldots & a_{n-1} - b_{n-1} & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & na_0 - b_0 & \ldots & \ldots & \ldots & a_{n-1} - b_{n-1} & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & na_0 - b_0 & (n-1)a_1 - b_1 & \ldots & \ldots & \ldots & a_{n-1} - b_{n-1} \end{pmatrix} \quad (3.33)$$

are zero.

4. Quantum case

4.1. **Characteristic identities.** Here we describe a general algorithm due to Gould [6] which produces a Cayley-Hamilton type identity for any M-element M_A in $\mathcal{Q}_\mu(\mathfrak{g})$, where $A \in Z(U(\mathfrak{gl}_n))$. We refer the reader to the original paper [6] for more details.

Let $\mu_1, \dots, \mu_l$ be all distinct weights appearing in the irreducible representation V_μ. Then, one can consider the following polynomial functions on $\mathfrak{h}$:

$$f_i(\lambda) = \frac{1}{2}(\chi_{\lambda+\mu_i}(A) - \chi_\lambda(A) - \chi_\mu(A)), \ \lambda \in \mathfrak{h}^*. \tag{4.1}$$

Here χ_λ is the infinitesimal character of $U(\mathfrak{g})$ that corresponds to a weight λ.

It is known that the twisted Weyl group $\widetilde{W}$ acts on $\{\mu_1, \dots, \mu_l\}$ and hence permutes functions $\{f_1, \dots, f_l\}$. Therefore, the coefficients $c_i(\lambda)$ of the polynomial

$$t^l + c_1(\lambda)t^{l-1} + \dots + c_l(\lambda) = (t - f_1(\lambda)) \dots (t - f_l(\lambda)) \tag{4.2}$$

are $\widetilde{W}$-invariant functions on $\mathfrak{h}^*$.

Recall that Harish-Chandra isomorphism $\Gamma\colon Z(U(\mathfrak{g})) \to S(\mathfrak{h})^{\widetilde{W}}$ acts as $(\Gamma(A))(\lambda) = \chi_\lambda(A)$ for $C \in Z(U(\mathfrak{g}))$. Gould proved the following fact.

Proposition 4.1. *For any $A \in Z(U)(\mathfrak{g})$ the polynomial*

$$t^l + \Gamma^{-1}(c_1(\lambda))t^{l-1} + \dots + \Gamma^{-1}(c_l(\lambda)) \tag{4.3}$$

annihilates the M-element $M_A \in \mathcal{Q}_\mu(\mathfrak{g})$.

Remark 4.1. The main difficulty in finding the explicit formulas of this polynomial is the computation of elements $\Gamma^{-1}(c_i(\lambda))$. That is because there is no convenient formula for the inverse of Harish-Chandra map $\Gamma^{-1}\colon S(\mathfrak{h})^{\widetilde{W}} \to Z(U(\mathfrak{g}))$.

4.2. ***M*-operator for the quadratic Casimir.** We prove here the analogue of Cayley-Hamilton identity. One can view these identities as generalizations of the well-known Capelli's identity. Namely, in case $\mathfrak{g} = \mathfrak{gl}_n$ and $\mu = \omega_1$ the row determinant

$$\operatorname{rdet}\begin{pmatrix} E_{11} - t & E_{21} & \dots & \dots & E_{n1} \\ E_{12} & E_{22} - t - 1 & \dots & \dots & E_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ E_{1n} & E_{2n} & \dots & \dots & E_{nn} - t - (n-1) \end{pmatrix}$$

annihilates M and this is closely related to the Capelli's identity.

It turns out that the characteristic identity of the M-operator of quadratic Casimir element in $\mathcal{Q}_{\omega_k}(\mathfrak{gl}_n)$ can be written in a way very similar to Cayley-Hamilton theorem. Define the diagonal matrix $Q_k \in \operatorname{Mat}(d(\omega_k), d(\omega_k)$ as follows:

$$Q_k = \operatorname{diag}([Q_k]_{II}, \ I \in \mathcal{I}_k), \ \text{where } [Q_k]_{II} = \sum_{i \in I} i - \frac{k(k+1)}{2}.$$

Now define the "correction" of M-element M_C as

$$\widehat{M_C} = M_C - Q_k.$$

Proposition 4.2. *The row determinant of the matrix $\widehat{M_C} - t \cdot \mathrm{Id}$ annihilates M_C, i.e.*

$$\operatorname{rdet}(\widehat{M_C} - t \cdot \mathrm{Id})|_{t=M_C} = 0. \tag{4.4}$$

Remark 4.2. In the classical case Cayley-Hamilton implies that $\det(M_C - t \cdot \mathrm{Id})|_{t=M_C} = 0$, so we do not have any corrections (i.e. Q_k). Thus, one can regard the identity above as a quantization of Cayley-Hamilton identity for M_C in the classical case.

Example 4.1. In the case $\mathfrak{g} = \mathfrak{gl}_4$ and $\mu = \omega_2$ we have the following matrix:

$$\widehat{M_C} = \begin{pmatrix} E_{11} + E_{22} & E_{32} & E_{42} & -E_{31} & -E_{41} & 0 \\ E_{23} & E_{11} + E_{33} - 1 & E_{43} & E_{21} & 0 & -E_{41} \\ E_{24} & E_{34} & E_{11} + E_{44} - 2 & 0 & E_{21} & E_{31} \\ -E_{13} & E_{12} & 0 & E_{22} + E_{33} - 2 & E_{43} & E_{32} \\ -E_{14} & 0 & E_{12} & E_{34} & E_{22} + E_{44} - 3 & E_{32} \\ 0 & -E_{14} & E_{13} & -E_{24} & E_{23} & E_{33} + E_{44} - 4 \end{pmatrix}$$

and the theorem states that the row determinant $\operatorname{rdet}(\widehat{M_C} - t \cdot \mathrm{Id})$ annihilates the element M_C. The correction Q_2 in this case is just the diagonal matrix $\operatorname{diag}(0, 1, 2, 2, 3, 4)$.

Proof. The proof relies on the following lemmas.

Lemma 4.3. *The coefficients of the polynomial* $\operatorname{rdet}(\widehat{M}_C - t\cdot \mathrm{Id})$ *belong to the center of the universal enveloping algebra* $U(\mathfrak{gl}_n)$.

Lemma 4.4. *The image of the polynomial* $\operatorname{rdet}(\widehat{M}_C - t\cdot \mathrm{Id}) \in U(\mathfrak{gl}_n)[t]$ *under the map* Γ *equals*

$$\prod_{I\in\mathcal{I}_k}\left(\sum_{i\in I} E_{ii} - t - [Q_k]_{II}\right) \in U(\mathfrak{h}).$$

Remark 4.3. Note that elements E_{ii} belong to $U(\mathfrak{h})$ and commute with each other. Hence, the product above is well-defined.

Proof. Denote for brevity $m = \binom{n}{k}$ and suppose that $I_1, \dots, I_m$ is the sequence of all elements of $\mathcal{I}_k$ in the increasing order (we use the ordering defined in Subsection 2.5). Let A_{IJ} be the (I, J)-entry of the matrix $\widehat{M}_C$. The definition of M_C implies that

$$M_C = \sum_{i,j=1}^{n} \pi_{\omega_k}(E_{ij}) \otimes E_{ji},$$

where $\pi_{\omega_k}(E_{ij})e_J = \sum_{I\in\mathcal{I}_k} \varepsilon_{j,i}(J, I)e_I$. Therefore,

$$A_{IJ} = \sum_{i,j=1}^{n} \varepsilon_{j,i}(J, I)E_{ji} - \delta_{IJ}(t + [Q_k]_{II}).$$

It follows that if $I \preceq J$ and $I \neq J$, then $A_{IJ} \in \mathfrak{n}_-$ and if $I = J$, then $A_{II} = \sum_{i\in I} E_{ii} - t - [Q_k]_{II} \in U(\mathfrak{h})$.

We have

$$\operatorname{rdet}(\widehat{M}_C - t\cdot \mathrm{Id}) = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma)A_{I_1I_{\sigma(1)}} \dots A_{I_mI_{\sigma(m)}}.$$

Now we claim that for any $\sigma \in \mathfrak{S}_m \setminus \{\mathrm{id}\}$ the product $A_{I_1I_{\sigma(1)}} \dots A_{I_mI_{\sigma(m)}}$ belongs to $\mathfrak{n}_-U(\mathfrak{gl}_n)$. Indeed, fix $\sigma \in \mathfrak{S}_m \setminus \{id\}$ and let $r \in \{1, 2, \dots, m\}$ be the minimal index such that $\sigma(r) \neq r$. Then, clearly, $\sigma(r) > r$ and $I_{\sigma(r)} \succeq I_r$. Thus, $A_{I_rI_{\sigma(r)}} \in \mathfrak{n}_-$ and

$$A_{I_1I_{\sigma(1)}} \dots A_{I_mI_{\sigma(m)}} = A_{I_1I_1} \dots A_{I_{r-1}I_{r-1}} \cdot A_{I_rI_{\sigma(r)}} \cdot \dots \cdot A_{I_mI_{\sigma(m)}} \in \mathfrak{n}_-U(\mathfrak{gl}_n).$$

This means that Γ maps each product $A_{I_1I_{\sigma(1)}} \dots A_{I_mI_{\sigma(m)}}$ with $\sigma \neq \mathrm{id}$ to zero. On the other hand, Γ maps $U(\mathfrak{h})$ to itself, so

$$\Gamma\left(\operatorname{rdet}(\widehat{M}_C - t\cdot \mathrm{Id})\right) = \prod_{I\in\mathcal{I}_k}\left(\sum_{i\in I} E_{ii} - t - [Q_k]_{II}\right),$$

as claimed. □

To finish the proof we need to check in the case of quadratic Casimir the functions $f_1, \dots, f_m \in S(\mathfrak{h})$ coincide with $\{\sum_{i\in I} E_{ii} - [Q_k]_{II}\}_{I\in\mathcal{I}_k} \subset U(\mathfrak{h})$ if we identify $S(\mathfrak{h})$ and $U(\mathfrak{h})$. This is the content of the following lemmas.

Lemma 4.5. *Let* $C = \sum_{i=1}^{m} X^iX_i$ *be the quadratic Casimir. Then,*

$$\chi_\lambda(C) = (\lambda, \lambda + 2\rho),$$

where ρ *is the half-sum of all positive roots of the reductive Lie algebra* $\mathfrak{g}$. *Here* $(\cdot,\cdot)$ *is the pairing on* $\mathfrak{h}^*$ *induced by invariant bilinear form* B *on* $\mathfrak{h}$.

Proof. See [6, Section 2]. □

Lemma 4.6. *Let* $\{\rho_i\}_{i=1}^n$ *be the basis in* $\mathfrak{h}^*$ *dual to basis* $\{E_{ii}\}_{i=1}^n$ *in* $\mathfrak{h}$. *Then,*

(a) *The set of all distinct weights of the irreducible representation* V_{ω_k} *is* $\{\mu_I\}_{I\in\mathcal{I}_k}$, *where* $\mu_I = \sum_{i\in I} \rho_i$.
(b) *In the case when* $A = C$ *is the quadratic Casimir element, the functions defined in Subsection 4.1 have the following explicit form:*

$$f_I(\lambda) = \sum_{i\in I}(\lambda, \rho_i) - \sum_{i\in I} i + \frac{k(k+1)}{2}. \tag{4.5}$$

Proof. The part (a) follows from the fact that $\{e_I\}_{I\in\mathcal{I}_k}$ is a basis of V_{ω_k} and it is not difficult to check that e_I is the vector of the weight μ_I.

For the part (b) we apply the formula (4.1):

$$f_I(\lambda) = \frac{1}{2}\big((\lambda + \mu_I, \lambda + \mu_I + 2\rho) - (\lambda, \lambda + 2\rho) - (\omega_k, \omega_k + 2\rho)\big).$$

Since $\mu_I = \sum_{i\in I}\rho_i$, $\omega_k = \sum_{i=1}^k \rho_i$, $\rho = \frac{1}{2}\sum_{i=1}^n (n+1-2i)\rho_i$ and $(\rho_i,\rho_j) = B(E_{ii},E_{jj}) = \delta_{ij}$, we have

$$f_I(\lambda) = \frac{1}{2}\big((2\lambda+2\rho+\mu_I,\mu_I) - (\omega_k,\omega_k+2\rho)\big) = (\lambda,\mu_I) + (\mu_I,\rho) - \omega_k,\rho) =$$

$$= \sum_{i\in I}(\lambda,\rho_i) + \frac{1}{2}\sum_{i\in I}(n+1-2i) - \frac{1}{2}\sum_{j=1}^k (n+1-2j) =$$

$$= \sum_{i\in I}(\lambda,\rho_i) - \sum_{i\in I} i + \frac{k(k+1)}{2},$$

as was claimed. □

Finally, under the identification of $S(\mathfrak{h})$ with $U(\mathfrak{h})$ the expression for f_I in the lemma above corresponds to $\sum_{i\in I} E_{ii} - [Q_k]_{II}$ and this concludes the proof. □

5. Conclusion

There are several questions related to Kirillov algebras which remain open. Firstly, it seems that the D-operator plays an important role in studying classical family algebras. In particular, using this operator one can construct commutative subalgebras inside $\mathcal{C}_\mu(\mathfrak{g})$. Since classical family algebras of certain classes, e.g. corresponding to weight multiplicity free representations, are isomorphic to equivariant cohomology rings of algebraic varieties, it might be interesting to understand the D-operator in terms of geometry.

Question 5.1. What is the geometrical meaning of the D-operator (at least in the weight multiplicity free case)?

On the algebraic side, we do not know any convenient way for the calculation of the action of D-operator in general case. Our results in case $\mathcal{C}_{\omega_k}(\mathfrak{gl}_n)$ were obtained by direct computations.

Question 5.2. Is there a natural coordinate-free description of the D-operator on arbitrary classical Kirillov algebra $\mathcal{C}_\mu(\mathfrak{g})$?

The study of quantum family algebras is much harder than in the classical case. In particular, even for $\mathcal{Q}_{\omega_k}(\mathfrak{gl}_n)$ we do not have explicit presentation of this algebra in terms of generators and relations (but we do know that $\mathcal{Q}_{\omega_k}(\mathfrak{gl}_n)$ is commutative). Using some non-trvial results from the theory of universal enveloping algebras we were able to find some identities for M-elements. However, it is unclear how one can get relations between different M-elements. The diference between classical and quantum cases here is that in the latter case we do not have an analogue of the “diagonalization map”.

Question 5.3. How to describe the quantum family algebra $\mathcal{Q}_{\omega_k}(\mathfrak{gl}_n)$ in terms of generators and relations?

Finally, one might expect that quantum family algebras correspond to some “quantized” versions of equivariant cohomology rings of certain algebraic varieties. However, here we do not even know what might be the suitable cohomology theory.

Question 5.4. Is there any geometrical description of quatum family algebras, e.g $\mathcal{Q}_{\omega_k}(\mathfrak{gl}_n)$?

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