Hausel's big algebras, commuting differential operators and Bethe subalgebras of the Yangian

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Kirillov algebras

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be a complex reductive Lie algebra. Let $\Pi\colon G \to \mathrm{GL}(V)$ be a direct sum of finite-dimensional representations of G, and denote by $\pi\colon \mathfrak{g} \to \mathrm{End}\, V$ the corresponding Lie algebra representation.

Definition

The **Kirillov algebra** of (Π, V) is defined as

$$\mathcal{C}(V) \coloneqq (S(\mathfrak{g}^*) \otimes \mathrm{End}\, V)^G.$$

More explicitly, the elements of the Kirillov algebras can be regarded as polynomial maps $F\colon \mathfrak{g} \to \mathrm{End}\, V$ which satisfy the following equivariance condition:

$$F(\mathrm{Ad}(g)(X)) = \Pi(g)F(X)\Pi(g)^{-1} \text{ for all } X \in \mathfrak{g},\ g \in G.$$

$\mathcal{C}(V)$ is a graded algebra over the ring of invariants $I(\mathfrak{g}) = S(\mathfrak{g}^*)^G$ with $I(\mathfrak{g})$ embedded into $\mathcal{C}(V)$ as a subalgebra of scalar operators.

A "quantized" version

The **Kostant algebra** (also known as the *strongly commuting ring*):

$$\mathcal{Q}(V) \coloneqq (U(\mathfrak{g}) \otimes \mathrm{End}\, V)^G.$$

Constructing elements of the Kirillov algebra

From the definition it is not quite clear how to construct elements of the Kirillov algebra. One can use a certain "differential-like" operator on $\mathcal{C}(V)$ to produce more elements of the Kirillov algebra.

Definition

Let $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$ and $\{X^i\}_{i=1}^{\dim \mathfrak{g}}$ be two bases of $\mathfrak{g}$ dual with respect to a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. For any $F \in \mathcal{C}(V)$ define the **Kirillov–Wei** operator $\mathbf{D} = \mathbf{D}_V : \mathcal{C}(V) \to \mathcal{C}(V)$ via

$$(\mathbf{D}F)(X) = \sum_{i=1}^{\dim \mathfrak{g}} \frac{\partial F}{\partial X_i}(X) \cdot \pi(X^i), \;\; X \in \mathfrak{g}.$$

Remark

For the quadratic invariant $F = \frac{1}{2}C \in I_2(\mathfrak{g})$ we get an example of a non-scalar element in $\mathcal{C}(V)$. In fact, we get $(\mathbf{D}F)(X) = \pi(X)$. In general, for any $P \in I(\mathfrak{g})$ the element $\mathbf{D}^k(P)$ belongs to the subalgebra $\langle S(\mathfrak{g}^*) \otimes \pi(\mathfrak{g})\rangle$, e.g.

$$\mathbf{D}^3(P) = \sum_{i,j,k=1}^{\dim \mathfrak{g}} \frac{\partial^3 F}{\partial X_i \partial X_j \partial X_k}(X) \cdot \pi(X^i)\pi(X^j)\pi(X^k).$$

A commutative subalgebra of the Kirillov algebra

The Kirillov algebra is not commutative in general. By experimenting on a computer Hausel and Zveryk found a way to construct a large commutative subalgebra of $\mathcal{C}(V)$ using the Kirillov–Wei operator.

Theorem (Hausel–Zveryk, 2022)

Let $\mathfrak{g}$ be simple Lie algebra of type A_n, B_n, C_n, D_n, or G_2. Then, there exists a set $\{c_1, \ldots, c_r\}$ of $r = \operatorname{rk}\mathfrak{g}$ homogeneous generators of $I(\mathfrak{g})$ satisfying the following property: for any representation V of $\mathfrak{g}$ the subalgebra

$$\mathcal{B}(V) = \langle \mathbf{D}^p(c_i) : i = 1, \ldots, r,\ p = 0, \ldots, \deg(c_i)\rangle$$

is a commutative subalgebra of $\mathcal{C}(V)$, and a free $I(\mathfrak{g})$-module of finite rank.

We call $\mathcal{B}(V)$ the **big algebra** associated to a representation V of $\mathfrak{g}$.

A more conceptual definition (for any simple Lie algebra)

The proof of Hausel–Zveryk relies on the existence of certain special complete sets of *Segal–Sugawara vectors* for the *Feigin–Frenkel center* $\mathfrak{z}(\hat{\mathfrak{g}})$. Big algebras can be defined as homomorphic images of the two-point *Gaudin subalgebra* in $(S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathfrak{g}}$:

$$\mathfrak{z}(\hat{\mathfrak{g}}) \hookrightarrow U(\mathfrak{g}_-) \xrightarrow{\phi_{z,\infty}} (S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathfrak{g}} \xrightarrow{\mathrm{id}\otimes\pi} (S(\mathfrak{g}) \otimes \operatorname{End} V)^{\mathfrak{g}} \simeq \mathcal{C}(V).$$

Medium algebras

The structure of the big algebra $\mathcal{B}(V)$ can be rather complicated in general. However, there is a smaller subalgebra of $\mathcal{B}(V)$ that is easier to understand.

Namely, one defines the **medium algebra** $\mathcal{M}(V)$ as the following subalgebra of $\mathcal{C}(V)$:

$$\mathcal{M}(V) = \langle \mathbf{D}^p(c_i) : i = 1, \ldots, r,\ p = 0, 1 \rangle = \langle I(\mathfrak{g}), \mathbf{D}(I(\mathfrak{g})) \rangle.$$

The medium algebra plays a special role because of the following property.

Theorem (Hausel, 2022)

Let V be an irreducible representation of $\mathfrak{g}$. Then, $\mathcal{M}(V)$ is the center of $\mathcal{C}(V)$.

Direct sums. Assume that $V = \bigoplus_i V_i$ is a direct sum of $\mathfrak{g}$-representations. It then follows from definitions that for each i there exist natural surjections

$$\mathcal{B}(V) \twoheadrightarrow \mathcal{B}(V_i), \quad \mathcal{M}(V) \twoheadrightarrow \mathcal{M}(V_i).$$

Some examples in the weight multiplicity free case

Usually it is not so easy to describe $\mathcal{B}(V)$ or even $\mathcal{M}(V)$ in terms of generators and relations. Here we give a few examples in the weight mutliplicity free case. For brevity we denote by $\mathcal{B}(\lambda)$ the big algebra of the irreducible representation $V(\lambda)$ of $\mathfrak{g}$ of highest weight λ (and similarly for $\mathcal{C}(\lambda)$ and $\mathcal{M}(\lambda)$).

Example 1 (minuscule case). if λ is a minuscule dominant weight, i.e. the set $\mathrm{wt}(V(\lambda))$ consists of a single W-orbit, then

$$\mathcal{B}(\lambda) = \mathcal{M}(\lambda) = \mathcal{C}(\lambda) \simeq \mathbb{C}[\mathfrak{h}]^{W_\lambda},$$

where W_λ is the stabilizer of $\lambda \in \mathfrak{h}^*$ in the Weyl group. For instance, if $\mathfrak{g} = \mathfrak{gl}_n$ and $\lambda = \varpi_k = (1, \dots, 1, 0 \dots, 0)$, then we obtain

$$\mathcal{B}(\varpi_k) = \mathcal{C}(\varpi_k) \simeq \mathbb{C}[t_1, \dots, t_n]^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} \text{ with } I(\mathfrak{gl}_n) \simeq \mathbb{C}[t_1, \dots, t_n]^{\mathfrak{S}_n}.$$

Example 2. Let $\mathfrak{g} = \mathfrak{gl}_n$ and $\lambda = m\varpi_1$, i.e. $V(\lambda) \simeq S^m(\mathbb{C}^n)$. In this case we get

$$\mathcal{B}(m\varpi_1) = \mathcal{M}(m\varpi_1) = \mathcal{C}(m\varpi_1) \simeq \mathbb{C}[t_1, \dots, t_n]^{\mathfrak{S}_n}[u_1, \dots, u_m]^{\mathfrak{S}_m} / I_{m\varpi_1}, \text{ where}$$

$$I_{m\varpi_1} = \left\langle \sum_{i=1}^{m} u_i^d (u_i - t_1) \dots (u_i - t_n) : 0 \leqslant d \leqslant m-1 \right\rangle, \; I(\mathfrak{gl}_n) \simeq \mathbb{C}[t_1, \dots, t_n]^{\mathfrak{S}_n}.$$

Algebro-geometric motivation: the geometry of the affine Grassmannian

Assume that $\mathfrak{g} = \mathfrak{gl}_n$ and $G = \mathrm{GL}_n$. To a dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_1 \geqslant \ldots \geqslant \lambda_n$, we associate the **affine Schubert variety**

$$\mathrm{Gr}^\lambda = \overline{\mathrm{GL}_n(\mathbb{C}[[z]]) \cdot z^\lambda \, \mathrm{GL}_n(\mathbb{C}[[z]])} \in \mathrm{Gr} = \mathrm{GL}_n(\mathbb{C}((z)))/\,\mathrm{GL}_n(\mathbb{C}[[z]]),$$

where $z^\lambda = \mathrm{diag}(z^{\lambda_1}, \ldots, z^{\lambda_n})$. For instance, for $\lambda = \varpi_k$ we get $\mathrm{Gr}^{\varpi_k} \simeq \mathrm{Gr}(k, n)$.

Theorem (Hausel, 2024; the case of $G = \mathrm{GL}_n$)

(i) The equivariant cohomology of Gr^λ is isomorphic to the corresponding medium algebra: $H^*_{\mathrm{GL}_n}(\mathrm{Gr}^\lambda) \simeq \mathcal{M}(\lambda)$ as modules over $H^*_{\mathrm{GL}_n}(\mathrm{pt}) \simeq I(\mathfrak{gl}_n)$.
(ii) The equivariant intersection cohomology of Gr^λ is isomorphic to the corresponding big algebra: $IH^*_{\mathrm{GL}_n}(\mathrm{Gr}^\lambda) \simeq \mathcal{B}(\lambda)$ as $\mathcal{M}(\lambda)$-modules.

In fact, in type A we have the following description of $\mathcal{M}(\lambda)$: if $\lambda_n \geqslant 0$ and $m = |\lambda|$, then

$$\mathcal{M}(\lambda) = R[u_1, \ldots, u_m]^{\mathfrak{S}_m}/I_\lambda \text{ with } R = I(\mathfrak{gl}_n) \simeq \mathbb{C}[t_1, \ldots, t_n]^{\mathfrak{S}_n},$$

where the ideal I_λ is given by

$$I_\lambda = \left\langle f \in R[u_1, \ldots, u_m]^{\mathfrak{S}_m} : f(t_{i_1}, \ldots, t_{i_m}) \equiv 0 \text{ whenever } \varepsilon_{i_1} + \ldots + \varepsilon_{i_m} \in \mathrm{wt}(\lambda) \right\rangle.$$

Two special GL_n-modules

Let $\{E_{ij}\}_{i,j=1}^n$ be the standard generators of $\mathfrak{g} = \mathfrak{gl}_n$ and let $\mathfrak{h} \subset \mathfrak{gl}_n$ be the Cartan subalgebra spanned by $\{E_{ii}\}_{i=1}^n$. We consider two models for $\mathfrak{gl}_n$-representations, namely $V_{n,r} = \mathbb{C}[\mathrm{Mat}_{n,r}] \simeq S^\bullet(\mathbb{C}^n \otimes \mathbb{C}^r)$ and $V'_{n,r} = \Lambda(\mathrm{Mat}_{n,r}) \simeq \wedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^r)$.

Denote by $\mathcal{PD}_{n,r}$ the non-commutative ring of (resp. odd) differential operators on $\mathrm{Mat}_{n,r}$ with polynomial coefficients. This ring has generators $\{x_{i\alpha}, \partial_{i\alpha}\}_{i,\alpha=1}^{n,r}$ (resp. $\{\xi_{i\alpha}, \partial_{i\alpha}\}_{i,\alpha=1}^{n,r}$) which are subject to relations

$$[x_{i\alpha}, x_{j\beta}] = [\partial_{i\alpha}, \partial_{j\beta}] = 0, \ [\partial_{i\alpha}, x_{j\beta}] = \delta_{ij}\delta_{\alpha\beta},$$
$$(\text{resp. } \{\xi_{i\alpha}, \partial_{j\beta}\} = \{\partial_{i\alpha}, \partial_{j\beta}\} = 0, \ \{\partial_{i\alpha}, \xi_{j\beta}\} = \delta_{ij}\delta_{\alpha\beta}).$$

The action of $U(\mathfrak{gl}_n)$ on $V_{n,r}$ (resp. $V'_{n,r}$) defined on generators by the formula

$$L\colon U(\mathfrak{gl}_n) \to \mathcal{PD}_{n,r}, \quad L(E_{ij}) = \sum_{\alpha=1}^r x_{i\alpha}\partial_{j\alpha} \left(\text{resp. } L(E_{ij}) = \sum_{\alpha=1}^r \xi_{i\alpha}\partial_{j\alpha}\right).$$

The **Howe duality** implies the following decompositions of $\mathrm{GL}_n \times \mathrm{GL}_r$-representations:

$$S^\bullet(\mathbb{C}^n \otimes \mathbb{C}^r) \simeq \bigoplus_{\lambda:\ell(\lambda)\leqslant \min\{n,r\}} V_{\mathrm{GL}_n}(\lambda) \otimes V_{\mathrm{GL}_r}(\lambda),$$
$$\wedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^r) \simeq \bigoplus_{\lambda:\ell(\lambda)\leqslant n, \ell(\lambda^t)\leqslant r} V_{\mathrm{GL}_n}(\lambda) \otimes V_{\mathrm{GL}_r}(\lambda^t).$$

Here partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ are interpreted as dominant weights for GL_n or GL_r.

Explicit formulas for the big algebra generators in type A

Let $\Phi_1, \dots, \Phi_n \in I(\mathfrak{gl}_n)$ be the coefficients of $\det(I_n + t \cdot Y)$ for $Y \in \mathfrak{gl}_n$.

Theorem (N.)

The big algebra $\mathcal{B}(V_{n,r})$ of $V_{n,r} = \mathbb{C}[\mathrm{Mat}_{n,r}]$ is generated by the operators $F_{p,q}$ for $p, q \geqslant 0$, $p + q \leqslant n$, defined as

$$F_{p,q}(Y) = \sum_{\substack{I_1, J_1 \in \binom{[n]}{p},\, I_2, J_2 \in \binom{[n]}{q} \\ I_1 \sqcup I_2 = J_1 \sqcup J_2}} \operatorname{sgn} \begin{pmatrix} I_1 & I_2 \\ J_1 & J_2 \end{pmatrix} \det Y_{I_1 J_1} \sum_{R \in \binom{[r]}{q}} \det(X_{J_2,R}) \det(D_{I_2,R}).$$

Moreover, the original generators of the big algebra related to these via

$$\mathbf{D}^q(\Phi_{p+q}) = q! \cdot F_{p,q} + (\mathbb{C}\text{-linear combination of } F_{p,s} \text{ with } s < q), \text{ while}$$

$$\sum_{\ell=0}^{n-p} (u(u-1)\dots(u-\ell+1))^{-1} \cdot F_{p,\ell}(Y) = (L \circ \mathrm{ev})(\sigma_{n-p}(u; Y^t)).$$

Here, $\mathrm{ev}\colon T(u) \mapsto 1 + u^{-1}E$ is the *evaluation homomorphism* from the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$ to $U(\mathfrak{gl}_n)$, and $\sigma_k(u; Y^t) = \frac{1}{n!} \mathrm{tr}_n(A_n T_1(u) \dots T_k(u-k+1) Y^t_{k+1} \dots Y^t_n)$ are the power series whose coefficients generate the *Bethe subalgebra* of $\mathrm{Y}(\mathfrak{gl}_n)$ with parameter Y^t. In particular, the operators $\{\mathbf{D}^q(\Phi_{p+q})\}_{p,q}$ commute with each other.

Simplifying the symmetrized determinants and permanents

Essentially the same can be done for $V'_{n,r} = \wedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^r)$. However, in this case one starts with the *permanent-type elements* in $\Psi_1, \dots, \Psi_n \in I(\mathfrak{gl}_n)$, i.e. the coefficients of the series $\det(I_n - z \cdot Y)^{-1} = \sum_{d \geqslant 0} \Psi_d(Y) \cdot z^d$ for $Y \in \mathfrak{gl}_n$. Then, the elements $\mathbf{D}^q(\Psi_{p+q})$ can be related to another set of generators of Bethe subalgebra.

The proof of the theorem is mostly computational. It relies on some variants of the *Capelli identity*, and a certain "cancellation lemma" which allows to get a manageable expression at the end.

Lemma (main calculation; $r = 1$)

(1) *Symmetric case:* for any k-tuple $I = (i_1, \dots, i_k)$ with distinct entries we have

$$\sum_{\sigma,\tau \in \mathfrak{S}_k} \operatorname{sgn}(\sigma\tau) x_{i_{\sigma(1)}} \partial_{i_{\tau(1)}} \dots x_{i_{\sigma(k)}} \partial_{i_{\tau(k)}} = (-1)^{k-1}(k-1)! \cdot \left(\sum_{s=1}^{k} x_{i_s} \partial_{i_s} \right).$$

(2) *Skew-symmetric case:* for any k-tuple $I = (i_1, \dots, i_k)$ we have

$$\sum_{\sigma,\tau \in \mathfrak{S}_k} \xi_{i_{\sigma(1)}} \partial_{i_{\tau(1)}} \dots \xi_{i_{\sigma(k)}} \partial_{i_{\tau(k)}} = (k-1)! \cdot \# \operatorname{Stab}(I) \cdot \left(\sum_{j=s}^{k} \xi_{i_s} \partial_{i_s} \right).$$

The expressions from the lemma arise naturally when one calculates $\mathbf{D}^q(\Phi_{p+q})$ and $\mathbf{D}^q(\Psi_{p+q})$.

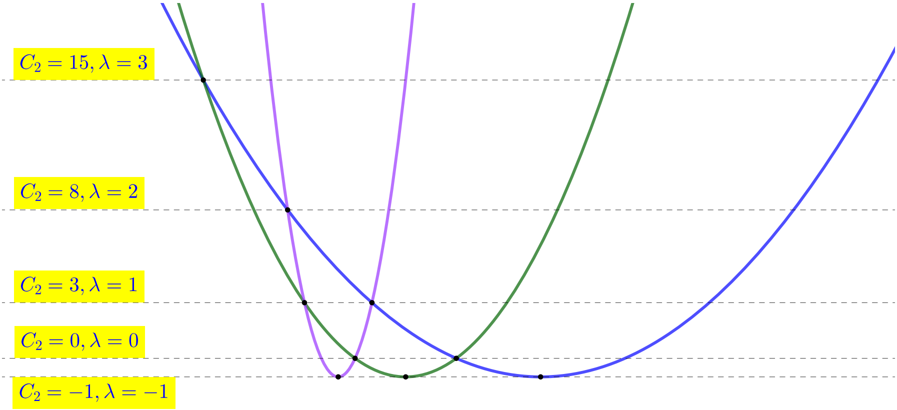
Summary and outlook

To sum up:

- to each representation V of a reductive Lie algebra $\mathfrak{g}$ we attach a commutative $I(\mathfrak{g})$-algebra $\mathcal{B}(V)$, called the **big algebra**. It is expected that one can "read off" a lot of representation-theoretic information by studying the fibers of the map $\operatorname{Spec}\mathcal{B}(V) \to \operatorname{Spec} I(\mathfrak{g})$;
- the original construction comes from the **Gaudin subalgebra** in $(S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathfrak{g}}$ which in turn can be related to the **Feigin–Frenkel center** $\mathfrak{z}(\hat{\mathfrak{g}})$;
- we suggest an alternative way to study type A big algebras; it allows us to reprove the commutativity of big algebras and relate them directly to **Bethe subalgebras** of the **Yangian**.

Further directions and ideas:

- other classical types: try to approach big algebras in types B, C, D by making use of the **twisted Yangians** and their Bethe subalgebras;
- Hausel also considers a version of a big algebra for the **Kostant algebra** defined as $(U(\mathfrak{g}) \otimes \operatorname{End} V)^{\mathfrak{g}}$. The spectrum of its center contains some information about the **category** $\mathcal{O}$ of $U(\mathfrak{g})$;
- **quantum groups and geometry:** the algebra $\langle Z_q, \Delta(Z_q)\rangle \subset U_q(\mathfrak{g}) \otimes \operatorname{End} V(\lambda)$ is conjecturally connected to the equivariant K-theory of the affine Schubert variety Gr^{λ} (Hausel & Löwit, work in progress).



$$\mathcal{Z}_{\mathfrak{sl}_2}(5\varpi_1) = Z\Big((U(\mathfrak{sl}_2) \otimes \operatorname{End} L(5\varpi_1))^{\mathfrak{sl}_2}\Big) \simeq$$

$$\mathbb{C}[C_2, M_1]/(M_1^2 + 20M_1 - 100C_2)(M_1^2 + 52M_1 - 36C_2 + 640)(M_1^2 + 68M_1 - 4C_2 + 1152)$$

Thank you!