Indecomposable characters on direct limit of groups $\mathfrak{S}_N \ltimes \mathbb{T}^N$ with diagonal embeddings

Nhok Tkhai Shon Ngo
(joint work with Mykola Nessonov)

Institute of Science and Technology Austria

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Characters on a group

Let G be a topological group. Recall the notion of a character on G:

Definition

A continuous function χ on a group G is called a *(normalized) character* if

- χ is positive-definite, i.e. the matrix $[\chi(g_j^{-1}g_k)]_{j,k=1}^m$ is positive semi-definite for any $g_1, g_2, \ldots, g_m \in G$,
- χ is central, i.e. the equality $\chi(gh) = \chi(hg)$ holds for all $g, h \in G$,
- $\chi(e) = 1$, where e is the identity element of G.

Definition

A character χ is called *indecomposable* if it is an extreme point of the simplex of all characters. In other words, χ is indecomposable if there are no distinct characters χ_1, χ_2 and $\alpha \in (0,1)$ such that $\chi = \alpha\chi_1 + (1-\alpha)\chi_2$.

Example

Multiplicative characters, i.e. group homomorphisms $\alpha\colon G \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ are characters in the sense above. More generally, the normalized trace given by $g \mapsto \frac{1}{\dim V} \mathrm{Tr}_V\, \pi(g)$ is also a character for any finite-dimensional representation V of G.

Block-diagonal embeddings and the inductive limit

Let positive integers N and M be such that N divides M. Define the *block-diagonal embedding* $\mathfrak{i} = \mathfrak{i}_{N,M} \colon \mathrm{U}(N) \hookrightarrow \mathrm{U}(M)$ as follows: for $U \in \mathrm{U}(N)$ we put

$$\mathfrak{i}_{N,M}(U) = U \otimes I_{M/N} \quad \text{or} \quad [U]_{N\times N} \overset{\mathfrak{i}_{N,M}}{\longmapsto} \begin{bmatrix} U & & & \\ & U & & \\ & & \ddots & \\ & & & U \end{bmatrix}_{M\times M}$$

Fix a sequence $\widehat{\mathbf{n}}$ of positive integers $\{n_j\}_{j=1}^{\infty}$ with $n_j > 1$, and set $N_k = n_1 n_2 \dots n_k$. For a suitable sequence of subgroups $\{H_k < \mathrm{U}(N_k)\}_k$ we can define the inductive limit

$$H_1 \overset{\mathfrak{i}_{N_1,N_2}}{\longrightarrow} H_2 \overset{\mathfrak{i}_{N_2,N_3}}{\longrightarrow} \dots \overset{\mathfrak{i}_{N_{k-1},N_k}}{\longrightarrow} H_k \overset{\mathfrak{i}_{N_k,N_{k+1}}}{\longrightarrow} \dots \longrightarrow \varinjlim H_k$$

Examples of resulting "infinite-dimensional groups" are given in the table below.

$H_k < \mathrm{U}(N_k)$	$\varinjlim H_k$
$\mathrm{U}(N_k)$	$\mathrm{U}(\widehat{\mathbf{n}})$
$\mathbb{T}^{N_k}$	$\mathbb{T}^{\widehat{\mathbf{n}}}$

$H_k < \mathrm{U}(N_k)$	$\varinjlim H_k$
$\mathfrak{S}_{N_k}$	$\mathfrak{S}_{\widehat{\mathbf{n}}}$
$G_{N_k} := \mathfrak{S}_{N_k} \ltimes \mathbb{T}^{N_k}$	$G_{\widehat{\mathbf{n}}}$

Goal: to classify all indecomposable characters on $G_{\widehat{\mathbf{n}}}$ (and then on $\mathrm{U}(\widehat{\mathbf{n}})$ as well).

Action on $L^2[0,1)$

The group $\mathfrak{S}_{\hat{\mathfrak{n}}}$ can be naturally identified with a subgroup of certain measure-preserving transformations of the space $X = [0,1)$ equipped with the Lebesgue measure ν.

Namely, we let an element σ of $\mathfrak{S}_{N_i} \subset \mathfrak{S}_{\hat{\mathfrak{n}}}$ to act on X as follows:

$$R(\sigma)(x) = \left(x - \frac{k-1}{N_i}\right) + \frac{\sigma(k)-1}{N_i}, \text{ if } x \in \left[\frac{k-1}{N_i}, \frac{k}{N_i}\right), \quad k = 1\ldots, N_i.$$

In other words, $\sigma \in \mathfrak{S}_{N_i}$ permutes N_i half-closed intervals $\left[\frac{k-1}{N_i}, \frac{k}{N_i}\right)$.

An element of the group $\mathbb{T}^{\hat{\mathfrak{n}}}$ can be regarded as measurable functions from $X = [0,1)$ to $\mathbb{T}$. Namely, we identify $\mathfrak{t} = (t_1, \ldots, t_{N_i}) \in \mathbb{T}^{N_i}$ with the following piecewise-constant function:

$$\mathfrak{t}(x) = t_k \text{ for } x \in \left[\frac{k-1}{N_i}, \frac{k}{N_i}\right), \quad k = 1\ldots, N_i.$$

Using these identifications we can embed $G_{\hat{\mathfrak{n}}}$ into the group of unitary operators of the Hilbert space $L^2(X,\nu)$. This gives a character $\chi(g) = \langle g \cdot \mathbb{1}_X, \mathbb{1}_X\rangle = \int_{\mathrm{Fix}(\sigma;X)} \mathfrak{t}(x)d\nu(x)$.

Basic characters

To state the main result, let us introduce certain special characters on $\mathfrak{S}_{\hat{\mathfrak{n}}}$ and $G_{\hat{\mathfrak{n}}}$.

Characters on $\mathfrak{S}_{\hat{\mathfrak{n}}}$: for any $\sigma \in \mathfrak{S}_{N_i} \subset \mathfrak{S}_{\hat{\mathfrak{n}}}$ define

$$\phi_{\mathrm{sym}}(\sigma; p) = \begin{cases} 1, & p = \mathrm{triv}, \\ \mathrm{sgn}_\infty(\sigma), & p = \mathrm{sgn}, \\ \chi_{\mathrm{reg}}(\sigma), & p = \mathrm{reg}, \\ \chi_{\mathrm{nat}}(\sigma), & p = \mathrm{nat}, \end{cases} \quad \text{where}$$

$$\chi_{\mathrm{nat}}(\sigma) = \frac{\#\{m \in \{1, 2, \ldots, N_i\} : \sigma(m) = m\}}{N_i} \quad \text{and} \quad \mathrm{sgn}_\infty(\sigma) = \lim_{k\to\infty} \mathrm{sgn}(\mathfrak{i}_{N_i, N_k}(\sigma)).$$

Characters on $G_{\hat{\mathfrak{n}}}$: for any r-tuple $\mathbf{m} = (M_1, \ldots, M_r) \in \mathbb{Z}^r$ and $g = (\mathfrak{t}, \sigma) \in G_{\hat{\mathfrak{n}}}$ set

$$\phi(g; M) = \int_{\mathrm{Fix}(\sigma; X)} \mathfrak{t}(x)^M d\nu(x) \quad \text{and} \quad \phi_{\mathbf{m}}(g) = \prod_{j=1}^{r} \phi(g; M_j).$$

In particular, the restriction of $\phi(-; 0)$ to $\mathfrak{S}_{\hat{\mathfrak{n}}}$ coincides with χ_{nat}.

Classification of indecomposable characters on $G_{\hat{\mathfrak{n}}}$

For $p \in \{\text{triv}, \text{sgn}, \text{reg}\}$ and $\mathbf{m} = (M_1, \dots, M_r) \in \mathbb{Z}^r$ we also denote

$$\phi_{p,\mathbf{m}}(g) = \phi_{\text{sym}}(\sigma; p)\phi_{\mathbf{m}}(g) = \begin{cases} \phi(g; M_1) \dots \phi(g; M_r), & p = \text{triv}, \\ \text{sgn}_\infty(\sigma)\phi(g; M_1) \dots \phi(g; M_r), & p = \text{sgn}, \\ \chi_{\text{reg}}(\sigma)\phi(g; M_1) \dots \phi(g; M_r), & p = \text{reg}. \end{cases}$$

Theorem 1 (Nessonov–N.)

Indecomposable characters on $G_{\hat{\mathfrak{n}}}$ are given by

$$\{\phi_{p,\mathbf{m}} : p \in \{\text{triv}, \text{sgn}, \text{reg}\}, \mathbf{m} \in \mathbb{Z}^r, r \in \mathbb{Z}_{\geqslant 0}\}.$$

Remark

One can check that $\phi_{p_1,\mathbf{m}_1} = \phi_{p_2,\mathbf{m}_2}$ iff $\phi_{\text{sym}}(-; p_1) = \phi_{\text{sym}}(-; p_2)$ and $\mathbf{m}_1$ coincides with $\mathbf{m}_2$ up to a permutation and deletion of zeroes if $p_1 = p_2 = \text{reg}$.

One can compare this result with the corresponding classification for $\mathfrak{S}_{\hat{\mathfrak{n}}}$.

Proposition (Nessonov-N., 2025)

Indecomposable characters of $\mathfrak{S}_{\hat{\mathfrak{n}}}$ are given by $\{\chi_{\text{reg}}, \chi^r_{\text{nat}}, \text{sgn}_\infty \cdot \chi^r_{\text{nat}} : r \in \mathbb{Z}_{\geqslant 0}\}$.

Outline of the proof of Theorem 1

Proof of the main theorem consists of three parts.

- We use the **approximation theorem**: each indecomposable character of $G_{\widehat{\mathbf{n}}}$ is a weak limit of some sequence of irreducible characters of groups G_{N_k}. This may be viewed as an instance of the *ergodic method* introduced by Vershik and Kerov.
- **Main technical part:** we describe all weakly convergent sequences of irreducible characters of groups G_{N_k} and compute their limits.
- We show that the resulting limiting functions are indeed extreme characters of $G_{\widehat{\mathbf{n}}}$.

Remark

- In the second part we also use the calculations from the proof of the classification of characters on $\mathfrak{S}_{\widehat{\mathbf{n}}}$.
- The third part is done essentially algebraically. We also give explicit constructions of unitary representations of $G_{\widehat{\mathbf{n}}}$ which realize $\phi_{p,\mathbf{m}}$ as matrix elements of the form $\langle \pi(g)\xi, \xi \rangle$.

General approximation theorem

Let $\{G(k)\}_{k=1}^{\infty}$ be a sequence of Hausdorff, second countable compact groups such that

$$G(1) \subset G(2) \subset \ldots \subset G(k) \subset \ldots$$

Let $G(\infty) = \bigcup_{k=1}^{\infty} G(k)$ be the inductive limit of the sequence $\{G(k)\}_{k=1}^{\infty}$. Denote by $\widehat{G}(k)$ the countable set of equivalence classes of all (finite-dimensional) irreducible unitary representations of the group $G(k)$.

Proposition

Let χ be an indecomposable character of the group $G(\infty)$. Then, there exists an increasing sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers and a sequence $\{\pi_{k_i} \in \widehat{G}(k_i)\}_{i=1}^{\infty}$ of irreducible representations of groups $G(k_i)$ such that the sequence $\{\chi_{\pi_{k_i}}\}_{i=1}^{\infty}$ of the corresponding characters converges to χ uniformly on each $G(m)$.

A sketch of proof (1/2)

- Apply the GNS construction for χ: it yields representation $(\Pi, \mathcal{H})$ with a cyclic unit vector ξ. The corresponding II$_1$-factor $M = \Pi(G(\infty))'' \subset B(\mathcal{H})$ has a tracial state defined as $\mathrm{tr}(m) = \langle m\xi, \xi\rangle_{\mathcal{H}}$. In particular, we have $\mathrm{tr}(\Pi(g)) = \chi(g)$ for $g \in G(\infty)$.

- Introduce the "conditional expectations": for $m \in M$ define

$$E_k(m) := \int_{G(k)} \Pi(g) m \Pi(g)^{-1} dg \in \Pi(G(k))'$$

and note that the limit $E_\infty(m) = \lim_{k\to\infty} E_k(m)$ exists for any m. Since M is a factor and $E_\infty(m) \in M \cap M'$, we get $E_\infty(m) = \mathrm{tr}(m) \cdot I$.

- Use a standard fact from the harmonic analysis on compact group $G(k)$: for any $h \in G(k)$ one has

$$\int_{G(k)} \Pi(g)\Pi(h)\Pi(g)^{-1} dg = \sum_{\pi \in \widehat{G}(n)} \chi_\pi(h) \cdot P_\pi,$$

where for $\pi \in \widehat{G}(k)$ the orthogonal projection P_π onto the π-isotypic component of Π is given by the formula

$$P_\pi = (\dim \pi)^2 \int_{G(n)} \chi_\pi(g^{-1})\Pi(g) dg.$$

A sketch of proof (2/2)

• Using the equality $\lim_{n\to\infty} \|E_n(m)\xi - E_\infty(m)\xi\|^2 = 0$ for $m = \int_{G(k)} f(h)\Pi(h)dh \in M$ with $f \in L^1(G(k))$, we obtain

$$\lim_{n\to\infty} \sum_{\pi\in\widehat{G}(n)} \left| \int_{G(k)} f(h)(\chi_\pi(h) - \chi(h))dh \right|^2 \cdot \|P_\pi\xi\|^2_{\mathcal{H}} = 0,$$

while $\sum_{\pi\in\widehat{G}(n)} \|P_\pi\xi\|^2_{\mathcal{H}} = \|\xi\|^2_{\mathcal{H}} = 1$ for each $n \geqslant k$.

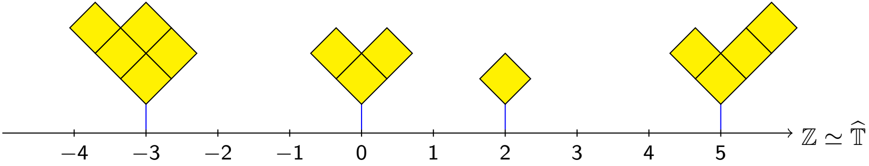
• Applying the fact above for countable dense subsets in each $L^1(G(k))$, we obtain a sequence $\pi_{k_i} \in \widehat{G}(k_i)$ such that $\chi_{\pi_{k_i}} \to \chi$ in weak-$*$ topology on each $L^\infty(G(k))$. Finally, for positive-definite functions the weak-$*$ topology is equivalent to the topology of uniform convergence on all compact subsets.

Irreducible representations of G_N

Each irreducible representation of the group G_N is isomorphic to

$$\pi_{\mathfrak{m},\rho} = \mathrm{Ind}_{\mathrm{Stab}(\mathfrak{m})\ltimes\mathbb{T}^N}^{G_N} \mathfrak{m}\otimes\rho,$$

where $\mathfrak{m}\in\mathbb{Z}^N$ is a multiplicative character of $\mathbb{T}^N$ and ρ is an irreducible representation of the stabilizer subgroup $\mathrm{Stab}(\mathfrak{m})\subset\mathfrak{S}_N$. Elements of $\widehat{G_N}$ can be parameterized by a function $\lambda\colon\widehat{\mathbb{T}}\to\mathbb{Y}$ with values in Young diagrams such that $|\lambda| = \sum_{k\in\mathbb{Z}}|\lambda(k)| = N$.



$\mathfrak{m} = (-3,-3,-3,-3,-3,0,0,0,2,5,5,5,5)$ $\rho = \rho_{(3,2)}\otimes\rho_{(2,1)}\otimes\rho_{(1)}\otimes\rho_{(2,1,1)}$

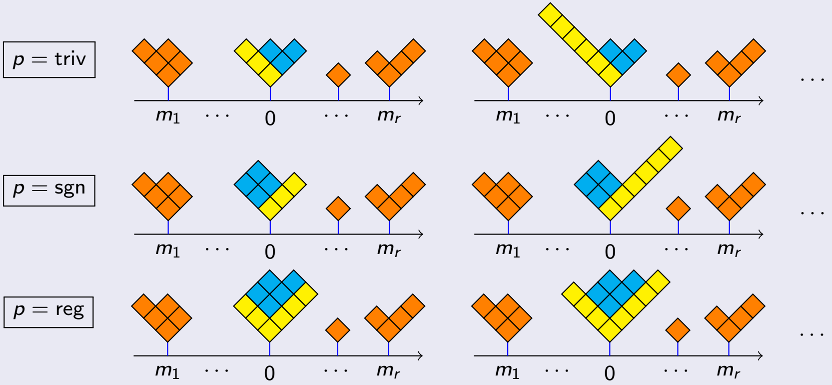
A function $\lambda\colon\mathbb{Z}\to\mathbb{Y}$ parameterizing an irreducible representation of $G_{13} = \mathfrak{S}_{13}\ltimes\mathbb{T}^{13}$.

We then use the Frobenius formula to express the character χ_λ of $\pi_{\mathfrak{m},\rho}$ in terms of characters of the symmetric groups.

Weakly converging sequences of characters of G_N

Lemma

Consider a sequence $\{\lambda(k)\colon \mathbb{Z} \to \mathbb{Y}\}_{k=1}^{\infty}$ with $|\lambda(k)| = N_k$. Assume that the sequence of irreducible characters $\chi_{\lambda(k)}$ converges pointwise on $G_{\widehat{\mathbf{n}}}$. Then, up to passing to a subsequence, the sequence $\{\lambda(k)\}_{k=1}^{\infty}$ is in one of the following "regimes":



The limiting function is $\phi_{p,\mathbf{m}}$, where $\mathbf{m}$ consists of $|\lambda(k)(m_i)|$ copies of m_i for $k \gg 1$ and of $\#\{$blue boxes$\}$ copies of 0 for $p = \mathrm{triv}, \mathrm{sgn}$ (no zeroes for $p = \mathrm{reg}$).

Sublemma

For $\mathfrak{t} = (t_1, \ldots, t_N) \in \mathbb{T}^N$ and $\mathfrak{m} = (m_1, \ldots, m_N)$ define

$$\psi(\mathfrak{t}; \mathfrak{m}) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} t_1^{m_{\sigma(1)}} \ldots t_N^{m_{\sigma(N)}}.$$

Assume that for a sequence $\{\mathfrak{m}_k \in \mathbb{Z}^{N_k}\}_k$ the functions $\psi(-; \mathfrak{m}_k)$ converge on $\mathbb{T}^{\hat{\mathfrak{n}}}$ to a continuous function. Then, there exist a finite collection $M_1, \ldots, M_r$ of integers such that each $\mathfrak{m}_k$, $k \gg 1$, coincides with $(M_1, \ldots, M_r, 0, \ldots, 0)$ up to a permutation.

Example

If $N_k = 2^k$, then the condition of the lemma in particular means that the sequence

$$\frac{t_1^{\mathfrak{m}_1(1)} t_2^{\mathfrak{m}_1(2)} + t_1^{\mathfrak{m}_1(2)} t_2^{\mathfrak{m}_1(1)}}{2}, \quad \frac{t_1^{\mathfrak{m}_2(1)+\mathfrak{m}_2(2)} t_2^{\mathfrak{m}_2(3)+\mathfrak{m}_2(4)} + t_1^{\mathfrak{m}_2(1)+\mathfrak{m}_2(3)} t_2^{\mathfrak{m}_2(1)+\mathfrak{m}_2(4)} + \ldots}{6}, \ \ldots$$

converges for all choices of $t_1, t_2 \in \mathbb{T}$.

"Explanations": (1) use the continuity at the identity and take elements of the form $(t, \ldots, t, 1, \ldots, 1)$; (2) if one manages to collect many terms of shape $t^m(t^k + t^{-k})$ or $t^m(t + t^{-1})^k$ with large $|k|$, we get a non-trivial bound on $\psi(-; \mathfrak{m}_k)$ and t around 1.

An application: characters of the infinite unitary group

Theorem 1 also allows us to recover the classification of characters for the group $\mathrm{U}(\widehat{\mathbf{n}})$.

Theorem 2 (Boyer 1993; Enomoto–Izumi, 2016)

The indecomposable characters of the group $\mathrm{U}(\widehat{\mathbf{n}}) = \varinjlim \mathrm{U}(N_k)$ are given by

$$\{\chi_{p,q} = \tau^p \overline{\tau}^q,\ p, q \in \mathbb{Z}_{\geqslant 0}\},$$

where τ is the character of $\mathrm{U}(\widehat{\mathbf{n}})$ given by the normalized trace, i.e. $\tau(u) = \frac{1}{N_k}\,\mathrm{tr}(u)$ for $u \in \mathrm{U}(N_k)$.

Indeed, one can restrict any given character $\chi\colon \mathrm{U}(\widehat{\mathbf{n}}) \to \mathbb{C}$ to the subgroup $G_{\widehat{\mathbf{n}}}$ to obtain that there exists a convex combination

$$\chi(u) = \sum_{\mathbf{m}} \alpha_{\mathbf{m}} \cdot \tau(u^{M_1}) \ldots \tau(u^{M_r}) \text{ for } u \in \mathrm{U}(\widehat{\mathbf{n}}), \text{ where } \alpha_{\mathbf{m}} \geqslant 0,\ \sum_{\mathbf{m}} \alpha_{\mathbf{m}} = 1.$$

To conclude the proof it remains to prove two facts:

- $\alpha_{\mathbf{m}}$ is 1 for some $\mathbf{m} = \mathbf{m}_0$ and is 0 for all others;
- for $\mathbf{m}_0 = (M_1, \ldots, M_r)$ all M_i's are ± 1.

Finishing the proof of Theorem 2

• To show that the convex combination above is trivial, we use the **multiplicativity property** of χ. In the homogeneous case, i.e. when $N_k = m^k$ for all k, this means that for any $u, v \in \mathrm{U}(\widehat{\mathfrak{n}})$ we have the equality

$$\chi(u \otimes v) = \chi(u)\chi(v).$$

To show that let $(\pi, \mathcal{H}, \xi)$ be the GNS representation for χ, then for any $u \in \mathrm{U}(m)$ there exists weak-$*$ limit

$$w-\lim_{k\to\infty} \pi(\underbrace{I_m \otimes \ldots \otimes I_m}_{k} \otimes u) = \chi(u) \cdot I.$$

The last equality holds because u and $I_m^{\otimes k} \otimes u$ are conjugate elements while $\pi(\mathrm{U}(\widehat{\mathfrak{n}}))''$ is a factor. Therefore,

$$\chi(u \otimes v) = \lim_{k\to\infty} \chi(u \otimes I_m^{\otimes k} \otimes v) = \lim_{k\to\infty} \langle \pi(u \otimes I_m^{\otimes k} \otimes v)\xi, \xi\rangle =$$
$$= \chi(v) \lim_{k\to\infty} \langle \pi(u)\xi, \xi\rangle = \chi(u)\chi(v).$$

The multiplicativity properties of $\chi(u)$ and $\tau(u^M)$, imply that $\alpha_{\mathbf{m}} = 0$ for all but one $\mathbf{m}$.

• To show that $\mathbf{m}_0 \in \{\pm 1\}^r$, we use the **representation theory** of $\mathrm{U}(N)$ and the theory of **symmetric functions**. Our claim then boils down to the following fact: the product of power sums $p_\lambda(t_1, \ldots, t_N)p_\mu(t_1^{-1}, \ldots, t_N^{-1})$ is a non-negative combination of "generalized Schur functions" iff $\lambda = (1^p)$ and $\mu = (1^q)$.

Thank you!